13.)

$$Z_3^2 \perp\!\!\!\perp e^{Z_1+4Z_2} \quad ?$$

Tehtävän 10 mukaan $\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}$ ja Z_3 ovat korreloimattomia eli $\mathrm{cov}\left(\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}, Z_3\right) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Koska (Z_1, Z_2, Z_3) on multinorm jakautunut

$\Rightarrow Z_3 \perp\!\!\!\perp \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \Rightarrow g(Z_3) \perp\!\!\!\perp h\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}$ kaikilla "kelvollisilla" g ja h

erityisesti $Z_3^2 \perp\!\!\!\perp e^{Z_1+4Z_2}$.

14) X, Y, Z kuten edellä

a) $W = \begin{pmatrix} Z_1 \\ Z_3 \end{pmatrix}$ jakauma?

b) $Z_2 \mid W=(w_1, w_2) \sim$?

a)

$(10) \Rightarrow Z \sim N\left(\begin{pmatrix} 0 \\ 7 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 10 \end{pmatrix}\right)$

$\Rightarrow W$ on myös multinorm. jakautunut

ja $EW = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\mathrm{Cov}(W) = \begin{pmatrix} 2 & 0 \\ 0 & 10 \end{pmatrix}$

b) Luennoista tiedämme

$$Y \mid (X=x) \sim N\left(\mu_Y + \Sigma_{YX}\Sigma_{XX}^{-1}(x-\mu_X),\ \Sigma_{YY} - \Sigma_{YX}\Sigma_{XX}^{-1}\Sigma_{XY}\right)$$

josta $\Rightarrow Z_2 \mid (W=(w_1,w_2))$ on (multi)norm. jakautunut

ja $E(Z_2 \mid W=(w_1,w_2)) = EZ_2 + \mathrm{cov}(\underset{1\times 1}{Z_2}, \underset{2\times 1}{W})(\mathrm{Cov}\, W)^{-1}\left(\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} - \underbrace{EW}_{=\binom{0}{0}}\right)$

$$= \underset{EZ_2}{7} + \underbrace{\begin{pmatrix} \mathrm{cov}(Z_2,Z_1) \\ \mathrm{cov}(Z_2,Z_3) \end{pmatrix}^T}_{\begin{pmatrix} 2 \\ 0 \end{pmatrix}^T = (2\ 0)} \underbrace{\begin{pmatrix} 2 & 0 \\ 0 & 10 \end{pmatrix}^{-1}}_{=\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{10} \end{pmatrix}} \cdot \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = 7 + (2\ 0)\begin{pmatrix} \frac{1}{2}w_1 \\ \frac{1}{10}w_2 \end{pmatrix}$$

$$= 7 + w_1$$

Ja $\quad \underset{1\times 1}{\mathrm{Cov}(Z_2} \mid W = \binom{w_1}{w_2}) = \mathrm{Var}\left(Z_2 \mid W = \binom{w_1}{w_2}\right)$

$$= \underbrace{\mathrm{Var}\, Z_2}_{=5} - \underbrace{\mathrm{cov}(Z_2, W)}_{=(2\ 0)} \underbrace{(\mathrm{Cov}\, W)^{-1}}_{=\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{10} \end{pmatrix}} \underbrace{\mathrm{cov}(W, Z_2)}_{\substack{= \mathrm{cov}(Z_2, W)^T \\ = \binom{2}{0}}}$$

$$= 5 - (2\ 0) \cdot \binom{1}{0} = 5 - 2 = 3$$

$\therefore \quad Z_2 \mid \left(W = \binom{w_1}{w_2}\right) \sim N(7 + w_1, 3)$.