1) $\mathbb{E}|X| = 4 \implies$ Markovin ey:tä voi soveltaa sille $|X|$

Markovin ey $\quad \mathbb{P}(|X| \geq a) \leq \dfrac{\mathbb{E}|X|}{a} = \dfrac{4}{a} \quad \forall a > 0$

a) $\quad \mathbb{P}(X \geq 32) \leq \mathbb{P}(|X| \geq 32) \leq \dfrac{4}{32} = \dfrac{1}{8}$

b) $\quad \mathbb{P}(X^2 > \underbrace{256}_{=2^8}) = \mathbb{P}(|X| > \underbrace{2^4}_{=16}) \leq \dfrac{4}{16} = \dfrac{1}{4}.$

2) $\operatorname{var} X = 16, \quad \mu = \mathbb{E}X. \quad 1 < \mu < 3$

Tšebyševin ey $\quad \mathbb{P}(|X - \mu| \geq t) \leq \dfrac{\operatorname{var} X}{t^2} = \dfrac{16}{t^2} \quad \forall t > 0$

Arvioidaan tapahtumaa $\{|X| > 9\}$.

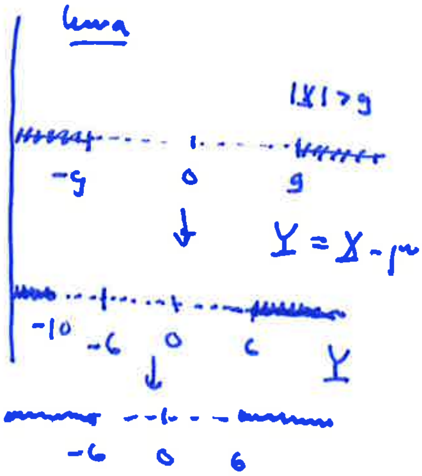
<u>Ol</u> $\quad |X| > 9 \iff X > 9$ tai $X < -9$

$\iff X - \mu > 9 - \mu$ tai $X - \mu < -9 - \mu$

$\implies X - \mu > 9 - \underbrace{\mu}_{<3} > 6$ tai $X - \mu < -9 - \underbrace{\mu}_{>1} < -10$

$\therefore \quad X - \mu > 6$ tai $X - \mu < -10$

$\implies |X - \mu| > 6$



sillä
- jos $X - \mu > 6 \implies |X - \mu| > 6$
- jos $X - \mu < -10 \implies |X - \mu| > 10 > 6$
  $\implies |X - \mu| > 6.$

$\therefore$

$\mathbb{P}(|X| > 9) \leq \mathbb{P}(|X - \mu| > 6)$

$\leq \mathbb{P}(|X - \mu| \geq 6) \leq \dfrac{16}{6^2} = \dfrac{16}{36} = \dfrac{2^4}{4 \cdot 9} = \underline{\dfrac{4}{9}}$

3) $\mathbb{P}(X \in (0,1)) = 1$, $\mathbb{E}X = 3/4$.

$\mathbb{E}\exp(2X) < \infty$

Kuvaus $g : (0,1) \to \mathbb{R}$, $g(x) = e^{2x}$ on konveksi

(sillä $\underbrace{D^2 e^{2x}}_{= 4e^{2x} > 0}$ on ~~konveksi~~ positiivinen)

Jensenin ey

$$\underbrace{g(\mathbb{E}X)}_{= g(\frac{3}{4}) = e^{\frac{6}{4}}} \le \mathbb{E}g(X)$$

$\therefore$ α:n min kän $e^{\frac{6}{4}} = e^{3/2} \approx 4{,}48$.

4) $f_{X,Y}(x,y) = cx^2 y \cdot \mathbb{1}\{0 < x < y \le 1\}$

a) $$\int_{\mathbb{R}^2} f_{X,Y}(x,y)\,dxdy = 1 \implies 1 = c\int_0^1 dy \int_0^y dx \; x^2 y$$

$$\int_0^1 dy \int_0^y dx \; x^2 y = \int_0^1 dy \Big/_0^y \frac{1}{3}x^3 = \frac{1}{3}\int_0^1 y \cdot y^3\,dy = \frac{1}{3}\int_0^1 y^4\,dy$$

$$= \frac{1}{3} \cdot \frac{1}{5} = \frac{1}{15}$$

$\therefore$ $1 = c \cdot \frac{1}{15} \implies c = 15.$

b) $$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy = 15x^2 \int_{-\infty}^{\infty} y \underbrace{\mathbb{1}\{0 < x < y < 1\}}_{= \mathbb{1}\{0 < x < 1\} \cdot \mathbb{1}\{x < y < 1\}} dy$$

$$= 15x^2 \mathbb{1}\{0 < x < 1\} \int_x^1 y\,dy$$

$$= 15x^2 \text{ — " — } \cdot \underbrace{\left(\Big/_x^1 \frac{1}{2}y^2 \right)}_{= \frac{1}{2}(1 - x^2)} = \frac{15}{2} \cdot x^2(1-x^2) \cdot \mathbb{1}\{0 < x < 1\}$$

c) $f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx$

$= \dots = 15 y\, \mathbb{1}\{0<y<1\} \int_0^y x^2\,dx$

$= 15 y\, \mathbb{1}\{0<y<1\}\, \frac{1}{3} y^3 = 5 y^4\, \mathbb{1}\{0<y<1\}.$

5) $Z = \begin{pmatrix} X \\ Y \end{pmatrix}$

a) $\mathbb{E}Z = \begin{pmatrix} \mathbb{E}X \\ \mathbb{E}Y \end{pmatrix}$

$\mathbb{E}X = \int_{-\infty}^{\infty} x f_X(x)\,dx \overset{4b)}{=} \int_0^1 \frac{15}{2} \underbrace{x \cdot x^2(1-x^2)}_{=x^3-x^5}\,dx$

$= \frac{15}{2} \Big/_0^1 \frac{x^4}{4} - \frac{1}{6}x^6$

$= \frac{15}{2}\left(\frac{1}{4} - \frac{1}{6}\right) = \frac{15}{\cancel{2}} \cdot \frac{\cancel{2}}{4\cdot 6} = \frac{15}{24} = \frac{5}{8}.$

$\mathbb{E}Y = \int_{-\infty}^{\infty} y f_Y(y)\,dy = 5\int_0^1 \underbrace{y y^4}_{=y^5}\,dy = 5 \Big/_0^1 \frac{1}{6} y^6 = \frac{5}{6}.$

$\therefore \quad \mathbb{E}Z = \begin{pmatrix} 5/8 \\ 5/6 \end{pmatrix}.$

b) $\mathrm{Cov}\, Z = \mathrm{cov}(Z,Z) = \mathbb{E}(Z-\mathbb{E}Z)(Z-\mathbb{E}Z)^T = \mathbb{E}ZZ^T - \mathbb{E}Z\,\mathbb{E}Z^T$

$= \begin{pmatrix} \mathrm{var}\, X & \mathrm{cov}(X,Y) \\ \mathrm{cov}(X,Y) & \mathrm{var}\, Y \end{pmatrix} = \begin{pmatrix} \mathbb{E}X^2 & \mathbb{E}(XY) \\ \mathbb{E}(XY) & \mathbb{E}Y^2 \end{pmatrix} - \begin{pmatrix} (\mathbb{E}X)^2 & \mathbb{E}X\,\mathbb{E}Y \\ \mathbb{E}X\,\mathbb{E}Y & (\mathbb{E}Y)^2 \end{pmatrix}$

↑ lasken nämä ↑ "nuo tiedossa"

$\mathbb{E}X^2 = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx = \frac{15}{2}\int_0^1 (x^4 - x^6)\,dx$

$= \frac{15}{2}\left(\frac{1}{5} - \frac{1}{7}\right) = \frac{15}{2} \cdot \frac{2}{5\cdot 7} = \frac{3}{7}$

$\Rightarrow \mathrm{var}\, X = \frac{3}{7} - \underbrace{(\mathbb{E}X)^2}_{= \frac{5^2}{2^6}}$

$= \frac{3}{7} - \frac{5^2}{2^6}$

$= \frac{2^6 \cdot 3 - 7 \cdot 5^2}{7 \cdot 2^6}$