1) $\mathbb{E}|X| = 2$ <u>MUISTUTUS</u>

a) $\mathbb{P}(X > 32) \le \mathbb{P}(|X| \geq 32) \le \frac{2}{32}$

b) $\mathbb{P}(X^2 > 256) = \mathbb{P}(|X| > 16) \le \frac{2}{16}$

4) a) $1 = \int_{\mathbb{R}^2} f_{X,Y}(x,y)\,dx\,dy = c \int_{\mathbb{R}^2} x^2 y \; \mathbb{1}\{0 \le x^2 < 2y < 1\}$.

$$= \begin{cases} \mathbb{1}\{0 \le |x| < 1,\ x^2 < 2y < 1\} \\ \mathbb{1}\{0 < 2y < 1,\ 0 \le x^2 < 2y\} \end{cases}$$

$$= \begin{cases} \mathbb{1}\{-1 < x < 1\}\,\mathbb{1}\{x^2 < 2y < 1\} \\ \mathbb{1}\{0 < y < \frac{1}{2},\ 0 \le |x| < \sqrt{2y}\} \end{cases}$$

$$= \begin{cases} \mathbb{1}\{-1 < x < 1\}\,\mathbb{1}\{\frac{1}{2}x^2 < y < \frac{1}{2}\} \\ \mathbb{1}\{0 < y < \frac{1}{2}\}\,\mathbb{1}\{-\sqrt{2y} < x < \sqrt{2y}\} \end{cases}$$

$\therefore$ KAKSI TAPAA INTEGROIDA

$$1 = c \int_{-1}^{1} dx \int_{\frac{1}{2}x^2}^{1/2} dy\; x^2 y = c \int_{-1}^{1} dx \left(x^2 \Big/_{\frac{1}{2}x^2}^{1/2} \frac{1}{2} y^2 \right)$$

$$= \frac{c}{2} \int_{-1}^{1} x^2 \underbrace{\left(\frac{1}{4} - \frac{1}{4}x^4 \right)}_{= \frac{1}{4}(1 - x^4)} dx \overset{\text{Sym.}}{=} \frac{c}{2} \cdot \frac{1}{4} \cdot 2 \int_0^1 x^2 (1 - x^4)\,dx$$

$$= \frac{c}{2} \cdot \frac{1}{4} \cdot 2 \int_0^1 (x^2 - x^6)\,dx = \frac{c}{8} \cdot 2 \cdot \Big/_0^1 \frac{1}{3}x^3 - \frac{1}{7}x^7$$

$$= \frac{c}{4} \cdot \left(\frac{1}{3} - \frac{1}{7} \right) = \frac{4c}{4 \cdot 7 \cdot 3} = \frac{c}{21}. \quad \Rightarrow \quad c = 21$$

TA1 $\quad 1 = c\int_0^{1/2} dy \int_{-\sqrt{2y}}^{\sqrt{2y}} dx\; x^2 y$

$$= 2c\int_0^{1/2} dy\left(y\int_0^{\sqrt{2y}} x^2\,dx\right) = 2c\int_0^{1/2} dy\left(y \Big/_0^{\sqrt{2y}} \tfrac{1}{3}x^3\right)$$

$$= \frac{2c}{3}\int_0^{1/2} \underbrace{y\; 2^{3/2}\, y^{3/2}}_{=y^{5/2}}\, dy = \frac{2c}{3}\cdot 2^{3/2} \Big/_0^{1/2} \frac{y^{7/2}}{7/2}$$

$$= \frac{2c}{3}\cdot \underbrace{2^{3/2}\cdot 2^{-7/2}}_{=2^{-4/2}}\cdot\frac{2}{7} = \frac{4c\cdot 2^{-2}}{3\cdot 7} = \frac{c}{21}. \qquad \therefore c = 21$$

b) $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy = c\int_{-\infty}^{\infty} \mathbb{1}\{-1<x<1\}\cdot \mathbb{1}\{\tfrac12 x^2<y<\tfrac12\}\; x^2 y\; dy$

$$= c x^2\, \mathbb{1}\{|x|<1\}\int_{\frac12 x^2}^{1/2} y\,dy$$

$$= \frac{cx^2}{2}\,\mathbb{1}\{|x|<1\}\left(\frac14 - \frac14 x^4\right) = \frac{cx^2}{8}(1-x^4)\,\mathbb{1}\{|x|<1\}$$

$$= \frac{21}{8}x^2(1-x^4)\;\mathbb{1}\{|x|<1\}.$$

c) $f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx = c\int_{-\infty}^{\infty} \mathbb{1}\{0<y<1/2\}\,\mathbb{1}\{|x|<\sqrt{2y}\}\; x^2 y\; dx$

Symm.

$$= 2cy\;\mathbb{1}\{0<y<1/2\}\int_0^{\sqrt{2y}} x^2\,dx = 2cy\;\mathbb{1}\{..\}\Big/_0^{\sqrt{2y}} x^3/3$$

$$= \frac{2\cdot 21\cdot y^{5/2}\cdot 2^{3/2}}{3}\cdot \mathbb{1}\{..\} = 2^{5/2}\cdot 7\,y^{5/2}\cdot \mathbb{1}\{0<y<\tfrac12\}$$

5) a) $\mathbb{E}Z = (\mathbb{E}X, \mathbb{E}Y)$

$$\mathbb{E}X = \int_{-\infty}^{\infty} x f_X(x)\,dx \underset{4b)}{=} \frac{21}{8}\int_{-1}^{1} \underbrace{x\cdot x^2(1-x^4)}_{\text{pariton}}\,dx = 0$$

$$\mathbb{E}Y = \int_{-\infty}^{\infty} y f_Y(y)\,dy \underset{4c)}{=} 2^{5/2}\cdot 7 \int_{0}^{1/2} \underbrace{y\cdot y^{5/2}}_{=y^{7/2}}\,dy$$

$$= 2^{5/2}\cdot 7\cdot \Big/_{0}^{1/2} \frac{y^{9/2}}{9/2} = \frac{2^{5/2}\cdot 2\cdot 7}{9}\cdot 2^{-9/2}$$

$$= \frac{2^{\overbrace{-4/2}^{=2}}\cdot 2\cdot 7}{9} = \frac{2^{-1}\cdot 7}{9} = \frac{7}{18}$$

$$\therefore\ \mathbb{E}Z = \left(0, \frac{7}{18}\right) = \begin{pmatrix} 0 \\ 7/18 \end{pmatrix}.$$

b) $$\operatorname{Cov} Z = \begin{pmatrix} \mathbb{E}X^2 & \mathbb{E}XY \\ \mathbb{E}XY & \mathbb{E}Y^2 \end{pmatrix} - \begin{pmatrix} (\overset{=0}{\mathbb{E}X})^2 & \overset{=0}{\mathbb{E}X}\,\mathbb{E}Y \\ 0 & 0 \end{pmatrix}$$

kuten edellisessä "rakn. ehdotuksessa"

$$\mathbb{E}X^2 \underset{\substack{4b)\\ +TTL}}{=} \frac{21}{8}\int_{-1}^{1} x^2\cdot x^2(1-x^4)\,dx \underset{\text{symm.}}{=} \frac{21}{8}\cdot 2\int_0^1 \underbrace{x^4(1-x^4)}_{=x^4-x^8}\,dx$$

$$= \frac{21}{4}\Big/_{0}^{1} \frac{1}{5}x^5 - \frac{1}{9}x^9$$

$$= \frac{21}{4}\left(\frac{1}{5}-\frac{1}{9}\right) = \frac{\cancel{21}^{\,7}\cdot \cancel{4}}{\cancel{4}\cdot 5\cdot \cancel{9}_{\,3}} = \frac{7}{15}$$

$$\mathbb{E}Y^2 \underset{\substack{4c)\\ +TTL}}{=} 2^{5/2}\cdot 7\int_0^{1/2} \underbrace{y^2\cdot y^{5/2}}_{=y^{9/2}}\,dy$$

$$= 2^{5/2}\cdot 7\cdot \Big/_{0}^{1/2} \frac{y^{11/2}}{11/2} = \frac{\overbrace{2\cdot 2^{5/2}}^{=2^{7/2}}\cdot 7\cdot 2^{-11/2}}{11} = \frac{7}{11\cdot 4}$$

$$\mathbb{E}XY \underset{\text{TM}}{=} c\int_0^{1/2} dy \left(\int_{-\sqrt{2y}}^{\sqrt{2y}} \underbrace{xy \cdot x^2 y}_{= x^3 y^2} \, dx \right) \underset{\text{paritet}}{=} c\int_0^{1/2} \underbrace{y^2 \cdot 0}_{=0} \, dy = \underline{0}$$

$$\therefore \quad \operatorname{Cov} Z = \begin{pmatrix} 7/15 & 0 \\ 0 & 7/44 \end{pmatrix}$$

6) a) $f_{Y|X}(y|x) = \dfrac{f_{X,Y}(x,y)}{f_X(x)} \underbrace{1\!\!1\{f_X(x) > 0\}}_{= 1\!\!1\{|x| < 1\}}$

$$= \begin{cases} \dfrac{\cancel{c}\cancel{x^2} y \, 1\!\!1\{|x|<1\} \, 1\!\!1\{\frac{1}{2}x^2 < y < \frac{1}{2}\}}{\frac{\cancel{c}\cancel{x^2}}{8}(1-x^4)} & \text{for } |x| < 1 \\ 0 & \text{ellers} \end{cases}$$

$$= \begin{cases} \dfrac{8y}{1-x^4} \, 1\!\!1\{\tfrac{1}{2}x^2 < y < \tfrac{1}{2}\} & \text{for } |x| < 1 \\ 0 & \text{ellers} \end{cases}$$

b) $f_{X|Y}(x|y) = \dfrac{f_{X,Y}(x,y)}{f_Y(y)} \underbrace{1\!\!1\{f_Y(y) > 0\}}_{= 1\!\!1\{0 < y < \frac{1}{2}\}}$

$$= \begin{cases} \dfrac{\cancel{c}x^2 y \, 1\!\!1\{0<y<\frac{1}{2}\} \, 1\!\!1\{|x| < \sqrt{2y}\}}{\frac{2\cancel{c}}{3} \cdot 2^{3/2} \cdot y^{5/2}} & \text{for } 0 < y < \frac{1}{2} \\ 0 & \text{ellers} \end{cases}$$

$$= \begin{cases} 3 \cdot 2^{-5/2} x^2 y^{-3/2} \, 1\!\!1\{|x| < \sqrt{2y}\} & \text{for } 0 < y < \frac{1}{2} \\ 0 & \text{ellers} \end{cases}$$

7) a) $\mathbb{E}(X \mid Y = y)$

$$= \int_{-\infty}^{\infty} x \, f_{X|Y}(x|y)\,dx \overset{6b)}{=} 1\{0<y<1/2\} \cdot 3 \cdot 2^{-5/2} \int_{-\infty}^{\infty} x^2 y^{-3/2} \cdot x \; 1\{|x|<\sqrt{2y}\}\,dx$$

$$= 3 \cdot 2^{-5/2} y^{-3/2} \cdot \underbrace{\int_{-\sqrt{2y}}^{\sqrt{2y}} x^3 \,dx}_{=0} \quad \text{für } 0<y<\tfrac{1}{2}$$

$$= 0$$

b) $\mathbb{E}(Y \mid X = x)$

$$= \int_{-\infty}^{\infty} y \, f_{Y|X}(y|x)\,dy \overset{6a)}{=} \frac{8}{1-x^4} 1\{|x|<1\} \cdot \underbrace{\int_{1/2x^2}^{1/2} y \cdot y \,dy}_{= \left[\frac{1}{3} y^3\right]_{1/2x^2}^{1/2} = \frac{1}{3}\left(\frac{1}{2}\right)^3 \cdot (1-x^6)}$$

$$= \frac{\cancel{8}(1-x^6)}{3(1-x^4)} \cdot \cancel{2^3} \; 1\{|x|<1\}$$

$$= \frac{1}{3} \frac{1-(x^2)^3}{(1-x^2)(1+x^2)} \; 1\{|x|<1\}$$

$$= \frac{1}{3} \frac{\cancel{(1-x^2)}(1+x^2+x^4)}{\cancel{(1-x^2)}(1+x^2)} \; 1\{|x|<1\} = \begin{cases} \frac{1}{3} \frac{1+x^2+x^4}{1+x^2} & \text{für } |x|<1 \\ 0 & \text{sonst} \end{cases}$$

9) $X \mid Y \sim N(Y, 4)$

$Y \sim Bin\left(6, \frac{3}{5}\right)$

a) Suoraan mallista $\mathbb{E}(X \mid Y) = Y$, sillä $X \mid Y$ noud. normaalijakaumaa jonka odotusarvo on Y ja varianssi 4.

b) $\mathbb{E}X = \mathbb{E}\,\mathbb{E}(X \mid Y) \overset{a)}{=} \mathbb{E}Y \overset{Y \sim Bin}{=} 6 \cdot \frac{3}{5} = \underline{\frac{18}{5}}$

d) Suoraan mallista $var(X \mid Y) = 4$ sillä ...

aivan kuin kohd. a)

c) $var\,\mathbb{E}(X \mid Y) \overset{a)}{=} var\,Y = 6 \cdot \frac{3}{5} \cdot \frac{2}{5} = \underline{\frac{36}{25}}$.

e) $var\,X \overset{\text{laskujen kaava}}{=} var\,\mathbb{E}(X \mid Y) + \mathbb{E}\,var(X \mid Y)$

$\overset{a)}{=} \frac{36}{25} + \mathbb{E}\underbrace{var(X \mid Y)}_{\overset{d)}{=}4} = \frac{36}{25} + 4 = \underline{\frac{136}{25}}$