Parabolic BMO and the forward-in-time maximal operator

Olli Saari, Aalto University

Singular integrals and partial differential equations
University of Helsinki

May 26, 2015
A function $u \in L^1_{loc}$ is said to be of bounded mean oscillation ($u \in \text{BMO}$) if

$$\|u\|_{\text{BMO}} = \sup_Q \int_Q |u - u_Q| < \infty.$$ 

The remarkable John-Nirenberg inequality asserts that

$$\sup_Q \int_Q \exp(\epsilon |u - u_Q|) < \infty$$

for some positive $\epsilon \lesssim \|u\|_{\text{BMO}}^{-1}$.

BMO is connected to many questions of harmonic analysis.

There is also an interesting connection between BMO and the regularity theory of elliptic PDE of divergence form.
A function $u \in L^1_{loc}$ is said to be of bounded mean oscillation ($u \in \text{BMO}$) if

$$\|u\|_{\text{BMO}} = \sup_Q \int_Q |u - u_Q| < \infty.$$ 

The remarkable John-Nirenberg inequality asserts that

$$\sup_Q \int_Q \exp(\epsilon |u - u_Q|) < \infty$$

for some positive $\epsilon \lesssim \|u\|_{\text{BMO}}^{-1}$.

BMO is connected to many questions of harmonic analysis.

There is also an interesting connection between BMO and the regularity theory of elliptic PDE of divergence form.
A function \( u \in L^1_{loc} \) is said to be of \textit{bounded mean oscillation} \((u \in \text{BMO})\) if

\[
\|u\|_{\text{BMO}} = \sup_Q \int_Q |u - u_Q| < \infty.
\]

The remarkable John-Nirenberg inequality asserts that

\[
\sup_Q \int_Q \exp(\epsilon |u - u_Q|) < \infty
\]

for some positive \( \epsilon \lesssim \|u\|^{-1}_{\text{BMO}} \).

BMO is connected to many questions of harmonic analysis.

There is also an interesting connection between BMO and the regularity theory of elliptic PDE of divergence form.
A function $u \in L^1_{loc}$ is said to be of \textit{bounded mean oscillation} ($u \in \text{BMO}$) if

$$\|u\|_{\text{BMO}} = \sup_Q \int |u - u_Q| < \infty.$$ 

The remarkable John-Nirenberg inequality asserts that

$$\sup_Q \int |u - u_Q| < \infty$$

for some positive $\epsilon \lesssim \|u\|_{\text{BMO}}^{-1}$.

BMO is connected to many questions of harmonic analysis.

There is also an interesting connection between BMO and the regularity theory of elliptic PDE of divergence form.
Let $A$ be a matrix of measurable functions $a_{ij}(x)$ such that

$$\Lambda^{-1}|\xi|^2 \leq \xi \cdot A\xi \leq \Lambda|\xi|^2$$

for some $\Lambda \in (1, \infty)$ uniformly in $x$.

If $w$ is a positive weak (super)solution to

$$\text{div}(A\nabla w) = 0 \quad \text{in } \Omega \subset \mathbb{R}^n,$$

then $u = \log w \in \text{BMO}(\Omega)$. This is an important observation in Moser’s proof of the DeGiorgi–Nash–Moser theorem.

As a consequence, $w^c \in A_2$. It is also true that $w \in A_1$.

Recall that $w \in A_p$ if

$$[w]_{A_p} = \sup_{Q \subset \Omega} \int_Q w \left( \int_Q w^{1-p'} \right)^{p-1} < \infty, \quad 1 \leq p \leq \infty.$$
Let $A$ be a matrix of measurable functions $a_{ij}(x)$ such that

$$\Lambda^{-1}|\xi|^2 \leq \xi \cdot A\xi \leq \Lambda|\xi|^2$$

for some $\Lambda \in (1, \infty)$ uniformly in $x$.

If $w$ is a positive weak (super)solution to

$$\text{div}(A \nabla w) = 0 \quad \text{in } \Omega \subset \mathbb{R}^n,$$

then $u = \log w \in \text{BMO}(\Omega)$. This is an important observation in Moser’s proof of the DeGiorgi–Nash–Moser theorem.

As a consequence, $w^c \in A_2$. It is also true that $w \in A_1$.

Recall that $w \in A_p$ if

$$[w]_{A_p} = \sup_{Q \subset \Omega} \int_Q w \left( \int_Q w^{1-p'} \right)^{p-1} < \infty, \quad 1 \leq p \leq \infty.$$
Let $A$ be a matrix of measurable functions $a_{ij}(x)$ such that

$$\Lambda^{-1} |\xi|^2 \leq \xi \cdot A\xi \leq \Lambda |\xi|^2$$

for some $\Lambda \in (1, \infty)$ uniformly in $x$.

If $w$ is a positive weak (super)solution to

$$\text{div}(A\nabla w) = 0 \quad \text{in } \Omega \subset \mathbb{R}^n,$$

then $u = \log w \in \text{BMO}(\Omega)$. This is an important observation in Moser’s proof of the DeGiorgi–Nash–Moser theorem.

As a consequence, $w^c \in A_2$. It is also true that $w \in A_1$.

Recall that $w \in A_p$ if

$$[w]_{A^p} = \sup_{Q \subset \Omega} \int_Q w \left( \int_Q w^{1-p'} \right)^{p-1} < \infty, \quad 1 \leq p \leq \infty.$$
Let $A$ be a matrix of measurable functions $a_{ij}(x)$ such that

$$\Lambda^{-1} |\xi|^2 \leq \xi \cdot A\xi \leq \Lambda |\xi|^2$$

for some $\Lambda \in (1, \infty)$ uniformly in $x$.

If $w$ is a positive weak (super)solution to

$$\text{div}(A\nabla w) = 0 \quad \text{in } \Omega \subset \mathbb{R}^n,$$

then $u = \log w \in \text{BMO}(\Omega)$. This is an important observation in Moser’s proof of the DeGiorgi–Nash–Moser theorem.

As a consequence, $w^c \in A_2$. It is also true that $w \in A_1$.

Recall that $w \in A_p$ if

$$[w]_{A_p} = \sup_{Q \subset \Omega} \int_Q w \left( \int_Q w^{1-p'} \right)^{p-1} < \infty, \quad 1 \leq p \leq \infty.$$
Let $A$ be a matrix of measurable functions $a_{ij}(x)$ such that

$$\Lambda^{-1}|\xi|^2 \leq \xi \cdot A\xi \leq \Lambda|\xi|^2$$

for some $\Lambda \in (1, \infty)$ uniformly in $x$.

If $w$ is a positive weak (super)solution to

$$\text{div}(A\nabla w) = 0 \quad \text{in } \Omega \subset \mathbb{R}^n,$$

then $u = \log w \in \text{BMO}(\Omega)$. This is an important observation in Moser’s proof of the DeGiorgi–Nash–Moser theorem.

As a consequence, $w^\epsilon \in A_2$. It is also true that $w \in A_1$.

Recall that $w \in A_p$ if

$$[w]_{A_p} = \sup_{Q \subset \Omega} \int_Q w \left( \int_Q w^{1-p'} \right)^{p-1} < \infty, \quad 1 \leq p \leq \infty.$$
We would like to study BMO in the context of parabolic differential equations.

We consider local solutions to e.g. one of the following

\[ u_t - \Delta u = 0, \]
\[ u_t - \text{div}(A \nabla u) = 0, \]
\[ (u^{p-1})_t - \text{div}(|\nabla u|^{p-2} \nabla u) = 0 \]

in \( \Omega \times (0, T) \). For our purposes, the last one is the most general one, and we will concentrate on it.

In general, the positive solutions cannot be Muckenhoupt \( A_2 \) weights in any obvious way (they can fail to be doubling measures with respect to any reasonable metric). Consequently, the parabolic BMO must be something non-trivial.
We would like to study BMO in the context of parabolic differential equations.

We consider local solutions to e.g. one of the following

\[ \begin{align*}
  u_t - \Delta u &= 0, \\
  u_t - \text{div}(A \nabla u) &= 0, \\
  (u^{p-1})_t - \text{div}(|\nabla u|^{p-2} \nabla u) &= 0
\end{align*} \]

in \( \Omega \times (0, T) \). For our purposes, the last one is the most general one, and we will concentrate on it.

In general, the positive solutions cannot be Muckenhoupt \( A_2 \) weights in any obvious way (they can fail to be doubling measures with respect to any reasonable metric). Consequently, the parabolic BMO must be something non-trivial.
We would like to study BMO in the context of parabolic differential equations.

We consider local solutions to e.g. one of the following

\[ u_t - \Delta u = 0, \]
\[ u_t - \text{div}(A \nabla u) = 0, \]
\[ (u^{p-1})_t - \text{div}(|\nabla u|^{p-2} \nabla u) = 0 \]

in \( \Omega \times (0, T) \). For our purposes, the last one is the most general one, and we will concentrate on it.

In general, the positive solutions cannot be Muckenhoupt \( A_2 \) weights in any obvious way (they can fail to be doubling measures with respect to any reasonable metric). Consequently, the parabolic BMO must be something non-trivial.
We would like to study BMO in the context of parabolic differential equations.

We consider local solutions to e.g. one of the following

\[ u_t - \Delta u = 0, \]
\[ u_t - \text{div}(A \nabla u) = 0, \]
\[ (u^{p-1})_t - \text{div}(|\nabla u|^{p-2} \nabla u) = 0 \]

in \( \Omega \times (0, T) \). For our purposes, the last one is the most general one, and we will concentrate on it.

In general, the positive solutions cannot be Muckenhoupt A_2 weights in any obvious way (they can fail to be doubling measures with respect to any reasonable metric). Consequently, the parabolic BMO must be something non-trivial.
I will give a summary of the recent results about parabolic BMO. Some of them are joint with J. Kinnunen. I will discuss

1. Notation in the space time $\mathbb{R}^{n+1}$.
2. The definition of parabolic BMO.
3. Weights.
4. The forward-in-time maximal operator.
The basic structure of $u_t - \Delta u = 0$ and its generalizations is preserved under translations $z \mapsto z + h$ and anisotropic dilations $(x, t) \mapsto (\delta x, \delta^p t)$ of the coordinates. ($p = 2$ for the heat equation)

These transformations generate parabolic rectangles. We denote

\[ R = R(x, t, L) = Q(x, L) \times (t - L^p, t + L^p), \]
\[ R^+(\gamma) = Q(x, L) \times (t + \gamma L^p, t + L^p) \quad \text{and} \]
\[ R^-(\gamma) = Q(x, L) \times (t - L^p, t - \gamma L^p). \]
The basic structure of \( u_t - \Delta u = 0 \) and its generalizations is preserved under translations \( z \mapsto z + h \) and anisotropic dilations \( (x, t) \mapsto (\delta x, \delta^p t) \) of the coordinates. (\( p = 2 \) for the heat equation)

These transformations generate parabolic rectangles. We denote

\[
R = R(x, t, L) = Q(x, L) \times (t - L^p, t + L^p),
\]

\[
R^+(\gamma) = Q(x, L) \times (t + \gamma L^p, t + L^p) \quad \text{and}
\]

\[
R^-(\gamma) = Q(x, L) \times (t - L^p, t - \gamma L^p).
\]
It was discovered in the 1960s that the solutions to parabolic equations $f$ satisfy

$$\int_{R^+(0)} \int_{R^-(0)} \sqrt{(u(x) - u(y))^+} \, dx \, dy < C(n, p)$$

for $u = -\log f$. (Moser, Trudinger)

The parabolic John-Nirenberg lemma (Moser, Trudinger, Aimar) tells that

$$\int_{R^+(\gamma)} \int_{R^-(-\gamma)} \exp(\epsilon(u(x) - u(y))^+) \, dx \, dy < C(n, p, \gamma)$$

for any $\gamma \in (0, 1)$. 
It was discovered in the 1960s that the solutions to parabolic equations $f$ satisfy

$$\int_{R^+(0)} \int_{R^-(0)} \sqrt{(u(x) - u(y))^+} \, dx \, dy < C(n, p)$$

for $u = -\log f$. (Moser, Trudinger)

The parabolic John-Nirenberg lemma (Moser, Trudinger, Aimar) tells that

$$\int_{R^+(\gamma)} \int_{R^-(\gamma)} \exp(\epsilon (u(x) - u(y))^+) \, dx \, dy < C(n, p, \gamma)$$

for any $\gamma \in (0, 1)$.
We define that \( u \in \text{PBMO}^- \) if
\[
\|u\|_{\text{PBMO}^-} := \sup_{R} \inf_{a} \left( \int_{R^{-}(\frac{1}{2})} (u - a)^+ + \int_{R^{+}(\frac{1}{2})} (a - u)^+ \right) < \infty.
\]

It holds that
\[
\|u\|_{\text{PBMO}^-} \approx_{n,p,\gamma} \sup_{R} \inf_{a} \left( \int_{R^{-}(\gamma)} (u - a)^+ + \int_{R^{+}(\gamma)} (a - u)^+ \right).
\]

It is possible to replace the constant \( a \) by a mean value in a certain cylinder:

\[
R^{-}(\gamma) \quad B \quad R^{+}(\gamma)
\]

\[
a_R = u_B
\]
We define that $u \in \text{PBMO}^-$ if

$$\|u\|_{\text{PBMO}^-} := \sup_R \inf_a \left( \int_{R^-\left(\frac{1}{2}\right)} (u - a)^+ + \int_{R^+\left(\frac{1}{2}\right)} (a - u)^+ \right) < \infty.$$ 

It holds that

$$\|u\|_{\text{PBMO}^-} \sim_{n,p,\gamma} \sup_R \inf_a \left( \int_{R^-\left(\gamma\right)} (u - a)^+ + \int_{R^+\left(\gamma\right)} (a - u)^+ \right).$$

It is possible to replace the constant $a$ by a mean value in a certain cylinder:

$R^-\left(\gamma\right)$ $B$ $R^+\left(\gamma\right)$

$$a_R = u_B$$
We define that $u \in \text{PBMO}^-$ if
\[
\|u\|_{\text{PBMO}^-} := \sup_R \inf_a \left( \int_{R^- \left(\frac{1}{2}\right)} (u - a)^+ + \int_{R^+ \left(\frac{1}{2}\right)} (a - u)^+ \right) < \infty.
\]

It holds that
\[
\|u\|_{\text{PBMO}^-} \sim_{n, p, \gamma} \sup_R \inf_a \left( \int_{R^- (\gamma)} (u - a)^+ + \int_{R^+ (\gamma)} (a - u)^+ \right).
\]

It is possible to replace the constant $a$ by a mean value in a certain cylinder:

\[
R^- (\gamma) \quad B \quad R^+ (\gamma)
\]

\[
a_R = u_B
\]
The weights $A_q^+(\gamma)$ corresponding to PBMO$^-$ are the ones satisfying

$$\sup \int_R \int_{R^{-}(\gamma)} w \left( \int_{R^{+}(\gamma)} w^{1-q'} \right)^{q-1} < \infty, \quad 1 < q < \infty.$$ 

As in the case of PBMO$^-$, we have that $A_q^+(\gamma) = A_q^+(\gamma')$ for all $\gamma, \gamma' \in (0,1)$.

It holds

$$\text{PBMO}^- = \{ \alpha \log w : w \in A_q^+, \alpha \in (0, \infty), q \in (1, \infty) \}.$$ 

(Kinnunen and S. 2014)
The weights $A^+_q(\gamma)$ corresponding to PBMO$^-$ are the ones satisfying

$$\sup_R \int_{R^-} w \left( \int_{R^+} w^{1-q'} \right)^{q-1} < \infty, \quad 1 < q < \infty.$$ 

As in the case of PBMO$^-$, we have that $A^+_q(\gamma) = A^+_q(\gamma')$ for all $\gamma, \gamma' \in (0,1)$.

It holds

$$\text{PBMO}^- = \{ \alpha \log w : w \in A^+_q, \alpha \in (0, \infty), q \in (1, \infty) \}.$$ 

(Kinnunen and S. 2014)
The weights $A_q^+(\gamma)$ corresponding to PBMO$^-$ are the ones satisfying

$$\sup_R \int_{R^-} w \left( \int_{R^+} w^{1-q'} \right)^{q-1} < \infty, \quad 1 < q < \infty.$$ 

As in the case of PBMO$^-$, we have that $A_q^+(\gamma) = A_q^+(\gamma')$ for all $\gamma, \gamma' \in (0, 1)$.

It holds

$$\text{PBMO}^- = \{ \alpha \log w : w \in A_q^+, \alpha \in (0, \infty), q \in (1, \infty) \}.$$ 

(Kinnunen and S. 2014)
We define the forward-in-time maximal function as

\[ M^{\gamma+} f(z) := \sup_{\ell > 0} \int_{R^+(z,\ell,\gamma)} |f|. \]

For \( q \in (1, \infty) \), the operator \( M^{\gamma+} : L^q(w) \to L^q(w) \) is bounded if and only if \( w \in A^+_q(\gamma) \) (Kinnunen and S. 2014).
We define the forward-in-time maximal function as

\[
M^{\gamma^+} f(z) := \sup_{\ell > 0} \int_{R^+(z,\ell,\gamma)} |f|.
\]

For \( q \in (1, \infty) \), the operator \( M^{\gamma^+} : L^q(w) \to L^q(w) \) is bounded if and only if \( w \in A^+_q(\gamma) \) (Kinnunnen and S. 2014).
**Theorem**

Let \( u \in \text{PBMO}^+ \) be non-negative. If \( M^+ u \in L^1_{\text{loc}} \), then \( M^+ u \in \text{PBMO}^+ \).

- The theorem holds true in \( \mathbb{R}^{n+1} \) and \( \Omega \times \mathbb{R} \).
- Nonnegativity is necessary at least in the latter case.
Theorem

Let $u \in \text{PBMO}^+$ be non-negative. If $M^\gamma u \in L^1_{\text{loc}}$, then $M^\gamma u \in \text{PBMO}^+$.

- The theorem holds true in $\mathbb{R}^{n+1}$ and $\Omega \times \mathbb{R}$.
- Nonnegativity is necessary at least in the latter case.
Theorem

Let $u \in \text{PBMO}^+$ be non-negative. If $M^\gamma u \in L^1_{\text{loc}}$, then $M^\gamma u \in \text{PBMO}^+$. 

- The theorem holds true in $\mathbb{R}^{n+1}$ and $\Omega \times \mathbb{R}$.
- Nonnegativity is necessary at least in the latter case.
Remarks

- It may be that the maximal function

$$M_{\gamma^+}^* u(z) = \sup_{R(z)} \left( (u^+)_{R^+}(\gamma) + (u^-)_{R^-}(\gamma) \right)$$

is more correct object than what was studied previously. They coincide for positive functions.

- $M_{\gamma^+}^*$ almost maps $\text{PBMO}^+ \to \text{PBMO}^+ + \text{PBMO}^-$ (compare to the classical $\text{BMO} \to \text{BMO}$).

- There exists a class of functions playing the role of “parabolic Hardy space $H^1$” in the sense of “duality”.
It may be the that the maximal function

\[ M_{\gamma}^+ u(z) = \sup_{R(z)} ((u^+)_{R^+}(\gamma) + (u^-)_{R^-}(\gamma)) \]

is more correct object than what was studied previously. They coincide for positive functions.

\[ M_{\gamma}^+ \text{ almost maps } \text{PBMO}^+ \to \text{PBMO}^+ + \text{PBMO}^- \text{ (compare to the classical BMO } \to \text{ BMO).} \]

There exists a class of functions playing the role of “parabolic Hardy space \( H^1 \)” in the sense of “duality”.
It may be the that the maximal function

$$M_{\ast}^{\gamma^+} u(z) = \sup_{R(z)} \left( (u^+)_{R^+}(\gamma) + (u^-)_{R^-}(\gamma) \right)$$

is more correct object than what was studied previously. They coincide for positive functions.

$M_{\ast}^{\gamma^+}$ almost maps $\text{PBMO}^+ \to \text{PBMO}^+ + \text{PBMO}^-$ (compare to the classical $\text{BMO} \to \text{BMO}$).

There exists a class of functions playing the role of “parabolic Hardy space $H^1$” in the sense of “duality”.