End point estimates and Monge–Ampère equation with drifts

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April 6, 2016
Stanislaw Ulam writes:
“One day in my office in North Hall of the University of Wisconsin, a young and brilliant graduate student named Richard Bellman appeared and expressed a desire to work with me.... I remembered that in Princeton Lefschetz had some new scientifcotechnological enterprise connected with the war efforts. I wrote to him about Bellman in a sort of Machiavellian way, saying that I had a very able student who was so good that he deserved considerable financial support, but I added that I doubt that Princeton could afford it. This immediately challenged Lefschetz, and he offered Bellman a position.... Two years later, Dick Bellman appeared in Los Alamos in uniform as a member of special engineering detachment....”
1. Main theorem

Theorem (Nazarov–Reznikov–Vasyunin–Volberg)

There exists an $A_1$ weight $w$ such that

$$
\|H : L^1(w) \to L^{1,\infty}(w)\| \geq c [w]_{A_1} \log^{1/4}(1 + [w]_{A_1})
$$

Let us fix the notation: $Q := [w]_{A_1} := \sup_x \frac{M_w(x)}{w(x)}$. Notice that $Q < \infty$ iff for every interval (cube) $I$, one has

$$
\langle w \rangle_I \leq C \inf_{x \in I} w(x).
$$

The smallest $C$ is $[w]_{A_1} =: Q \geq 1$.

In other words, for any sufficiently large $Q$ one can find a weight $w$, a function $f$, and a number $\lambda > 0$ such that

$$
w\{x : Hf(x) > \lambda\} \geq c\lambda^{-1} Q \log^{1/4} Q \int |f(x)| w(x) dx. \quad (1)
$$
2. A brief history

Muckenhoupt 40 years ago posed two problems:
1) prove (or disprove) that
\[ w\{x : Hf(x) > \lambda \} \leq c\lambda^{-1} \int |f(x)|Mw(x)\,dx. \quad (2) \]

2) If this inequality is correct, then for any \( w \in A_1 \), with \( Q = [w]_{A_1} \) one will have automatically
\[ w\{x : Hf(x) > \lambda \} \leq c\lambda^{-1}Q \int |f(x)|w(x)\,dx. \quad (3) \]

Suppose inequality (2) is incorrect, then prove (or disprove) (3).
There can be 3 possible answers: a) (2) is correct, b) (2) fails, but (3) holds (in other words, there is no counterexample for “smooth” weights), c) (3) fails. Obvious: if (3) fails then (2) fails. But there is no other obvious claim.
3. A brief history

Maria Reguera and Christoph Thiele disproved (2) in 2009. That was a sophisticated counterexample, but the weight $w$ was very much irregular, and very far from being from $A_1$. So the so-called “weak Muckenhoupt conjecture” or $A_1$-conjecture was still open:

$$w \in A_1 \Rightarrow w\{ x : Hf(x) > \lambda \} \leq c \lambda^{-1} Q \int |f(x)| w(x) dx \quad ??? \ (4)$$

As a Theorem on slide 1 or (1) shows, weak Muckenhoupt conjecture gets also disproved: the claim above is false, and one can detect a logarithmic blow-up—see $\log^{1/4} Q$ in (1) on slide 1.

**What is known for the estimate from above for**

$\| H : L^1(w) \to L^{1,\infty}(w) \|$ for $[w]_{A_1} = Q < \infty$, $Q >> 1$?

**Theorem (Lerner–Ombrosi–Pérez)**

$$w\{ x : Hf(x) > \lambda \} \leq c \lambda^{-1} Q \log Q \int |f(x)| w(x) dx . \quad (5)$$
4. Dyadic singular operators first

Our measure space throughout this article will be \((X, \mathcal{A}, dx)\), where \(\sigma\)-algebra \(\mathcal{A}\) is generated by a standard dyadic filtration \(\mathcal{D} = \bigcup_k \mathcal{D}_k\) on \(\mathbb{R}\). We consider the martingale transform (and the square function transform) related to this homogeneous dyadic filtration. For our case of dyadic lattice on the line we have that \(|\Delta_J f|\) is constant on \(J\), and

\[
\Delta_J f = \frac{1}{2} [ (\langle f \rangle_J^+ - \langle f \rangle_J^-) 1_J^+ + (\langle f \rangle_J^- - \langle f \rangle_J^+) 1_J^- ].
\]

The square function transform: \((S \varphi)^2(x) = \sum_{J \in \mathcal{D}} |\Delta_J \varphi|^2 1_J(x)\). Recall that the martingale transform is the operator given by \((|\varepsilon_J| \leq 1)\):

\[
T \varphi = \sum_{J \in cD} \varepsilon_J \Delta_J \varphi.
\]

\[
\frac{1}{|I|} \mathbb{w}\{x \in I : \sum_{J \in \mathcal{D}(I)} \varepsilon_J (\varphi, h_J) h_J(x) > \lambda\} \leq C_{[w]_{A_1}} \frac{\langle |\varphi| w \rangle_I}{\lambda}.
\]

\(6\)
5. Results for Martingale Transform

Theorem (NRVV)

There is a positive absolute constant $c$ and a weight $w \in A_1$ such that constant $C_{[w]_{A_1}}$ from (6) satisfies

$$C_{[w]_{A_1}} \geq c[w]_{A_1} (\log[w]_{A_1})^{1/4}.$$ 

Theorem (LOP)

For any weight $w \in A_1$ constant $C_{[w]_{A_1}}$ from (6) satisfies

$$C_{[w]_{A_1}} \leq c[w]_{A_1} \log[w]_{A_1}.$$ 

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To find the “some estimates on” $C_{[w]_{A_1}}$ we use again the Bellman function technique. The idea is to reformulate the infinitely dimensional problem of optimization of $C_{[w]_{A_1}}$, that is finding of the “smallest” $C_{[w]_{A_1}}$ that works for all inequalities (6), in terms of the growth estimate on a certain function of only finite number of variables (5 in this case). Here it is. It will depend on number $Q \geq 1$.

$$B(F, w, m, f, \lambda) := B_Q(F, w, m, f, \lambda) := \sup \frac{1}{|I|} \omega \{ x \in I : \sum_{J \subseteq I, J \in D} \varepsilon_J(\varphi, h_J)h_J(x) > \lambda \},$$

(7)

where the sup is taken over all $\varepsilon_J, |\varepsilon_J| \leq 1, J \in D(I)$, and over all $\varphi \in L^1(I, \omega \, dx)$ such that $F := \langle |\varphi| \omega \rangle_I$, $f := \langle \varphi \rangle_I$, $w = \langle \omega \rangle_I$, $m \leq \inf_I \omega$, and $\omega$ are all dyadic $A_1$ weights, such that $[w]_{A_1} \leq Q$. 
This function is obviously defined in the convex subdomain of \( \mathbb{R}^5 \):

\[
\Omega := \{ (F, w, m, f, \lambda) \in \mathbb{R}^5 : F \geq |f| m, \ m \leq w \leq Q m \}. \tag{8}
\]

\[
sB \left( \frac{F}{s}, \frac{w}{s}, \frac{m}{s}, f, \lambda \right) = B(F, w, m, f, \lambda),
\]

\[
B(tF, w, m, tf, t\lambda) = B(F, w, m, f, \lambda).
\]

Introducing new variables \( \alpha = \frac{F}{m\lambda}, \beta = \frac{w}{m}, \gamma = \frac{f}{\lambda} \) we can see that

\[
\frac{1}{m} B(F, w, m, f, \lambda) = B\left( \frac{F}{m\lambda}, \frac{w}{m}, \frac{f}{\lambda} \right) =: B(\alpha, \beta, \gamma), \tag{9}
\]

where function \( B(\alpha, \beta, \gamma) = B(\alpha, \beta, 1, \gamma, 1) \). \( B \) is defined in the domain

\[
G := \{ (\alpha, \beta, \gamma) : |\gamma| \leq \alpha, 1 \leq \beta \leq Q \}. \tag{10}
\]
8. Properties of $\mathcal{B}_Q$: a special form of concavity

**Theorem**

Let $P, P_+, P_- \in \Omega$, $P = (F, w, \min(m_+, m_-), f, \lambda)$,
$P_+ = (F + A, w + u, m_+, f + a, \lambda + ta)$,
$P_- = (F - A, w - u, m_-, f - a, \lambda - ta)$, $0 \leq t \leq 1$. Then

$$B(P) - \frac{1}{2}(B(P_+) + B(P_-)) \geq 0. \quad (11)$$

At the same time, if
$P, P_+, P_- \in \Omega$, $P = (F, w, \min(m_+, m_-), f, \lambda)$,
$P_+ = (F + A, w + u, m_+, f + a, \lambda - ta)$,
$P_- = (F - A, w - u, m_-, f - a, \lambda + ta)$, $0 \leq t \leq 1$. Then

$$B(P) - \frac{1}{2}(B(P_+) + B(P_-)) \geq 0. \quad (12)$$

In particular $B(\alpha, \beta, \gamma)$ of slide 7 is concave: just put $t = 0$ here.
9. Properties of $\mathcal{B}_Q$: a special form of concavity

In particular, with fixed $m$, and with all points being inside $\Omega$ we get for all $t \in [0, 1]$

$$\mathcal{B}(F, w, m, f, \lambda) \geq \frac{1}{4} \left( \mathcal{B}(F - dF, w - dw, m, f - d\lambda, \lambda - td\lambda) + \mathcal{B}(F - dF, w - dw, m, f + d\lambda, \lambda - td\lambda) + \mathcal{B}(F + dF, w + dw, m, f - d\lambda, \lambda + td\lambda) + \mathcal{B}(F + dF, w + dw, m, f + d\lambda, \lambda + td\lambda) \right).$$

(13)

In fact, only $t = 0$ and $t = 1$ should be looked upon. Let us look at $t = 1$ case. In lines one and four $f_+ - f_- = \lambda_+ - \lambda_-$. In lines two and three $f_+ - f_- = -(\lambda_+ - \lambda_-)$. In both case $|f_+ - f_-| = |\lambda_+ - \lambda_-|$. 

Remark

1) Differential notation $dF, dw, d\lambda$ just mean small numbers, 2) in (13) we loose a bit of information (in comparison with (11),(12)), but this is exactly (13) that we are going to use in the future.
10. Sketch of the proof

Fix $P, P_+, P_- \in \Omega$. Let $\varphi_+, \varphi_-, \omega_+, \omega_-$ be functions and weights giving the supremum in $\mathcal{B}(P_+), \mathcal{B}(P_-)$ respectively up to a small number $\eta > 0$. Using the fact that $\mathcal{B}$ does not depend on $I$, we think that $\varphi_+, \omega_+$ is on $I_+$ and $\varphi_-, \omega_-$ is on $I_-$. Consider

$$\varphi(x) := \begin{cases} \varphi_+(x), & x \in I_+ \\ \varphi_-(x), & x \in I_- \end{cases}; \quad \omega(x) := \begin{cases} \omega_+(x), & x \in I_+ \\ \omega_-(x), & x \in I_- \end{cases}$$

Put $a := \Delta_I \varphi = \frac{1}{2}(P_{+,4} - P_{-,4})$. Notice that for $x \in I_+, \varepsilon_I = -t$,

$$\frac{1}{|I|} \omega_+ \{ x \in I_+ : \sum_{J \subseteq I_+, J \in D} \varepsilon_J(\varphi, h_J)h_J(x) > \lambda \} =$$

$$\frac{1}{|I|} \omega_+ \{ x \in I_+ : \sum_{J \subseteq I_+, J \in D} \varepsilon_J(\varphi, h_J)h_J(x) > \lambda + ta \}$$

$$= \frac{1}{2|I_+|} \omega_+ \{ x \in I_+ : \sum_{J \subseteq I_+, J \in D} \varepsilon_J(\varphi_+, h_J)h_J(x) > P_{+,5} \} \geq \frac{1}{2} \mathcal{B}(P_+) - \eta.$$
11. Sketch of the proof

Similarly, for \( x \in I_- \) we get if \( \epsilon_I = -t, \, 0 \leq t \leq 1 \),

\[
\frac{1}{|I|} \omega_- \{ x \in I_- : \sum_{J \subseteq I, J \in D} \epsilon_J(\varphi, h_J)h_J(x) > \lambda \} = \frac{1}{|I|} \omega_- \{ x \in I_- : \sum_{J \subseteq I, J \in D} \epsilon_J(\varphi, h_J)h_J(x) > \lambda - ta \} = \frac{1}{2|I_-|} \omega_- \{ x \in I_- : \sum_{J \subseteq I_- , J \in D} \epsilon_J(\varphi_-, h_J)h_J(x) > P_- , 5 \} \geq \frac{1}{2} B(P_-) - \eta .
\]

Combining the two left hand sides we obtain for \( \epsilon_I = -1 \)

\[
\frac{1}{|I|} \omega \{ x \in I_+ : \sum_{J \subseteq I, J \in D} \epsilon_J(\varphi, h_J)h_J(x) > \lambda \} \geq \frac{1}{2} (B(P_+) + B(P_-)) - 2\eta .
\]
12. Sketch of the proof

Obviously $P_3 = \min(P_{3,-}, P_{3,+}) = \min(\min_{I_-} \omega_-, \min_{I_+} \omega_+)$, $P_5 = \lambda$,

$$\langle \varphi \rangle_I = F = P_1, \langle \omega \rangle_I = w = P_2, \langle \varphi \rangle_I = f = P_4. \quad (14)$$

Let us use now the simple information (14): if we take the supremum in the left hand side over all functions $\varphi$, such that $\langle \varphi \rangle_I = F, \langle \varphi \rangle_I = f, \langle \omega \rangle_I = w$, and weights $\omega$: $\langle \omega \rangle_I = w$, in dyadic $A_1$ with $A_1$-norm at most $Q$, and supremum over all $\varepsilon_J = \pm s, s \in [0, 1]$, (only $\varepsilon_I = -1$ stays fixed), we get a quantity smaller or equal than the one, where we have the supremum over all functions $\varphi$, such that $\langle \varphi \rangle_I = F, \langle \varphi \rangle_I = f, \langle \omega \rangle_I = w$, and weights $\omega$: $\langle \omega \rangle_I = w$, in dyadic $A_1$ with $A_1$-norm at most $Q$, and an unrestricted supremum over all $\varepsilon_J = \pm s, s \in [0, 1], \varepsilon_I = -t, 0 \leq t \leq 1$. The latter quantity is of course $B(F, w, m, f, \lambda)$. So we proved (11).

To prove (12) we repeat verbatim the same reasoning, only keeping now $\varepsilon_I = t, 0 \leq t \leq 1$. We are done with “fancy concavity” proof.
13. Property in $m$: function $t \rightarrow \frac{1}{t} B(t\alpha, t\beta, \gamma)$ is increasing

Function $B$ is obviously decreasing in $m$. In fact, if $m$ decreases (all other coordinates vein fixed) then the collection of weights increases, and the supremum increases. It is not difficult to see that $B$ is also continuous.

$$\frac{1}{m} B(F, w, m, f, \lambda) = B\left(\frac{F}{m\lambda}, \frac{w}{m}, \frac{f}{\lambda}\right) =: B(\alpha/m, \beta/m, \gamma), \quad (15)$$

So $t \rightarrow \frac{1}{t} B(t\alpha, t\beta, \gamma)$ is increasing.
14. Two more properties, domain and symmetry

It is easy to see from the definition of $B$ that it is even in its variable $f$. Therefore,

$$B(\alpha, \beta, \gamma) = B(\alpha, \beta, -\gamma).$$

Notice that the concavity of $B$ (in $\gamma$) and this symmetry together imply that $\gamma \to B(\cdot, \cdot, \gamma)$ is decreasing on $\gamma \in [0, \alpha]$. The domain of definition of $B$ is

$$G_Q := \{(\alpha, \beta, \gamma) \in \mathbb{R}^3 : 1 \leq \beta \leq Q, |\gamma| \leq \alpha\}.$$

For function with all these properties the following holds.

**Theorem**

*There are absolute positive constant $c$ such that for some point $(\alpha, \beta, \gamma) \in G$

$$B(\alpha, \beta, \gamma) \geq cQ(\log Q)^{1/4}\alpha.$$* (16)
Now a couple of words about the idea of the proof of Theorem of slide 14. Ideally we would like to find the formula for $B$ (and therefore for $B$ because of (15)). To proceed we rewrite the second property of $B$ as a PDE on $B$. Then we try to find the boundary conditions on $B$ on $\partial G$, and then we may hope to solve this PDE. Unfortunately there are many roadblocks on this path, starting with the fact that the second property of $B$ is not a PDE, it is rather a partial differential inequality in discrete form. We will write it down as a pointwise partial differential inequality, but for that we will need a subtle result of Aleksandrov. We also can find boundary values of $B$, see some of them in next slides. However, the main difficulty is that our partial differential expression is in $3D$. 

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16. Unweighted case

We first consider the simplest case of \( m = \omega = 1 \) identically. The we are left with function \( Bel(F, f, \lambda) = B(F, 1, 1, f, \lambda) \), which is defined in a convex domain \( \Omega_0 \subset \mathbb{R}^3 \):
\[
\Omega_0 := \{(F, f, \lambda) \in \mathbb{R}^3 : |f| \leq F \},
\]
and whose concavity properties are described in

**Theorem**

Let \( P, P_+, P_- \in \Omega_0, P = (F, f, \lambda), P_+ = (F + A, f + a, \lambda + ta), P_- = (F - A, f - a, \lambda - ta), t \in [0, 1]. \) Then

\[
Bel(P) - \frac{1}{2}(Bel(P_+) + Bel(P_-)) \geq 0. \tag{17}
\]

At the same time, if \( P, P_+, P_- \in \Omega_0, P = (F, f, \lambda), P_+ = (F + A, f + a, \lambda - ta), P_- = (F - A, f - a, \lambda + ta), t \in [0, 1]. \) Then

\[
Bel(P) - \frac{1}{2}(Bel(P_+) + Bel(P_-)) \geq 0. \tag{18}
\]
Let us make the change of variables, \((F, f, \lambda) \rightarrow (F, y_1, y_2)\):

\[
y_1 := \frac{1}{2}(\lambda + f), \quad y_2 := \frac{1}{2}(\lambda - f).
\]

Denote

\[
M(F, y_1, y_2) := B(F, y_1 - y_2, y_1 + y_2) = Bel(F, f, \lambda).
\]

In terms of function \(M\):

**Theorem**

*The function \(M\) is defined in the domain \(G := \{(F, y_1, y_2) : |y_1 - y_2| \leq F\}\), and for each fixed \(y_2\), \(M(F, y_1, y_2)\) is concave in \((F, y_1)\) and for each fixed \(y_1\), \(M(F, y_1, y_2)\) is concave in \((F, y_2)\).*

The properties of \(M\) remind strongly the properties of Burkholder function.
18. Unweighted case

In the unweighted situation we can find \( B \) (or \( M \)) precisely.

**Theorem**

\[
\beta_{el}(F, f, \lambda) = \begin{cases} 
1, & \text{if } \lambda \leq F, \\
1 - \frac{(\lambda - F)^2}{\lambda^2 - f^2}, & \text{if } \lambda > F.
\end{cases}
\]  

(19)

This result means that we found a boundary value of the Bellman function \( B(F, w, m, f, \lambda) \) of the weighted problem on the part of its boundary, namely we found this function of 5 variables on \( \{ P \in \partial\Omega : w = P_2 = P_3 = m \} \).

\[
B(F, m, m, f, \lambda) = m \begin{cases} 
1, & \text{if } \lambda \leq F, \\
1 - \frac{(\lambda - F)^2}{\lambda^2 - f^2}, & \text{if } \lambda > F.
\end{cases}
\]  

(20)

Thus, the boundary values of \( B \):

\[
B(\alpha, 1, \gamma) = \begin{cases} 
1, & \text{if } \alpha \geq 1, \\
1 - \frac{(1-\alpha)^2}{1-\gamma^2}, & \text{if } 0 \leq |\gamma| \leq \alpha < 1.
\end{cases}
\]  

(21)
Let $\mathcal{B}el_0(F, f, \lambda)$ = the same function as on slide 11 but $\varepsilon_I$ are allowed to be only $\pm 1$.

**Theorem**

$$\mathcal{B}el_0(F, f, \lambda) = \begin{cases} 1, & \text{if } \lambda \leq F, \\ 1 - \frac{(\lambda - F)^2}{\lambda^2 - f^2}, & \text{if } \lambda > F \end{cases} = \mathcal{B}el(F, f, \lambda) \quad (22)$$

By definition $\mathcal{B}el_0 \leq \mathcal{B}el$: $\varepsilon_I = \pm 1$ versus $\varepsilon_I \in [-1, 1]$. In Banach space norm such martingale transforms obviously have the same norm. But we work now with $L^{1, \infty}$. By Sten–Weiss lemma

$$\|MT_{[-1,1]}\|_{L^{1,\infty}} \leq 2(2 + \log 2 \sum k2^{-k})\|MT_{\pm 1}\|_{L^{1,\infty}}.$$  

But we got from the Theorem above that the norms are equal:

$$\|MT_{[-1,1]}\|_{L^{1,\infty}} = \|MT_{\pm 1}\|_{L^{1,\infty}}.$$  

How to get this equality without the use of Bellman functions?
We can mollify $\mathcal{B}$ to make it smooth and still to have its “fancy concavity properties”. But then we loose homogeneity, and cannot reduce $\mathcal{B}$ to $\mathcal{B}$. We can mollify $\mathcal{B}$ to keep its homogeneity—just choose the mollifier depending on the point—but then we lose its “fancy concavity property”. In short, we have a problem with the mollification. This is why Aleksandrov’s theorem is very useful now.
We saw on slide 8 that $b$ is concave. By the result of Aleksandrov, $B$ has all second derivatives almost everywhere, this means that for a. e. $x \in G^\circ$ and all small vectors $h \in \mathbb{R}^3$,

$$B(x + h) = B(x) + \nabla B(x) \cdot h + \langle H_B(x) \cdot h, h \rangle + o(|h|^2), \quad (23)$$

where $H_B$ is the Hessian matrix of $B$. On the other hand, the “fancy concavity property” of slide 9 can be rewritten in terms of $B$ as follows: $B(\frac{F}{\lambda}, \beta, \frac{f}{\lambda})$—

$$\frac{1}{4} \left[ B(\frac{F - dF}{\lambda - d\lambda}, \beta - d\beta, \frac{f - d\lambda}{\lambda - d\lambda}) + B(\frac{F - dF}{\lambda - d\lambda}, \beta - d\beta, \frac{f + d\lambda}{\lambda - d\lambda}) + \right.$$  

$$\left. B(\frac{F + dF}{\lambda + d\lambda}, \beta + d\beta, \frac{f - d\lambda}{\lambda + d\lambda}) + B(\frac{F + dF}{\lambda + d\lambda}, \beta + d\beta, \frac{f + d\lambda}{\lambda + d\lambda}) \right] \geq 0. \quad (24)$$
21. From discrete inequality to differential inequality via Aleksandrov’s theorem

**Theorem**

For almost every point \( P = (\alpha, \beta, \gamma) =: \left( \frac{F}{\lambda}, \beta, \frac{f}{\lambda} \right) \in G^o \) and every vector \((dF, d\beta, d\lambda) \in \mathbb{R}^3\) we have

\[
- \alpha^2 B_{\alpha \alpha}(P) \left( \frac{dF}{F} - \frac{d\lambda}{\lambda} \right)^2 - \beta^2 B_{\beta \beta}(P) \left( \frac{d\beta}{\beta} \right)^2 - (1 + \gamma^2) B_{\gamma \gamma}(P) \left( \frac{d\lambda}{\lambda} \right)^2 - 2\alpha \beta B_{\alpha \beta}(P) \left( \frac{dF}{F} - \frac{d\lambda}{\lambda} \right) \frac{d\beta}{\beta} + 2\beta \gamma B_{\beta \gamma}(P) \frac{d\beta}{\beta} \frac{d\lambda}{\lambda} + 2\alpha \gamma B_{\alpha \gamma}(P) \left( \frac{dF}{F} - \frac{d\lambda}{\lambda} \right) \frac{d\lambda}{\lambda} + 2\alpha B_{\alpha}(P) \left( \frac{dF}{F} - \frac{d\lambda}{\lambda} \right) \frac{d\lambda}{\lambda} - 2\gamma B_{\gamma}(P) \left( \frac{d\lambda}{\lambda} \right)^2 \geq 0. \tag{25}
\]
Let us call by $\mathcal{N}$ the matrix of the quadratic form in (25). After a rather straightforward operation $\mathcal{N} \rightarrow \mathcal{M} := A^* \mathcal{N} A$ with an invertible matrix $A$ we can write down the non-negativity of the differential form in (25) as the a.e. in $G^\circ$ non-negativity of the following matrix

$$\mathcal{M}_1 := \begin{bmatrix}
-\alpha^2 B_{\alpha\alpha}, & -\alpha\beta B_{\alpha\beta}, & \alpha\gamma B_{\alpha\gamma} + \alpha B_{\alpha} \\
-\alpha\beta B_{\alpha\beta}, & -\beta^2 B_{\beta\beta}, & \beta\gamma B_{\beta\gamma} \\
\alpha\gamma B_{\alpha\gamma} + \alpha B_{\alpha}, & \beta\gamma B_{\beta\gamma}, & -(1 + \gamma^2) B_{\gamma\gamma} - 2\gamma B_{\gamma} \\
\end{bmatrix} \geq 0.$$  

(26)

$$\mathcal{M}_2 := \begin{bmatrix}
-\alpha^2 B_{\alpha\alpha}, & -\alpha\beta B_{\alpha\beta}, & -\alpha\gamma B_{\alpha\gamma} \\
-\alpha\beta B_{\alpha\beta}, & -\beta^2 B_{\beta\beta}, & -\beta\gamma B_{\beta\gamma} \\
-\alpha\gamma B_{\alpha\gamma}, & -\beta\gamma B_{\beta\gamma}, & -\gamma^2 B_{\gamma\gamma} \\
\end{bmatrix} \geq 0.$$  

(27)
Taking half-sum of (26) and (27), we obtain the following non-negativity:

\[ M := \begin{bmatrix} -\alpha^2 B_{\alpha\alpha} & -\alpha \beta B_{\alpha\beta} & \frac{1}{2} \alpha B_\alpha \\ -\alpha \beta B_{\alpha\beta} & -\beta^2 B_{\beta\beta} & 0 \\ \frac{1}{2} \alpha B_\alpha & 0 & -(\frac{1}{2} + \gamma^2) B_{\gamma\gamma} - \gamma B_\gamma \end{bmatrix} \geq 0. \] (28)

It is now natural to restrict the quadratic form of this matrix on certain 2D hyperplanes in the 3D tangent space \( Tan_p \) of the graph \( \Gamma := \{ p := (P, B(P)), P \in G^\circ \} \) at a given point \( p \). Namely, let us consider the quadratic form of matrix \( M \) in (26) on vectors of the form

\[ (\xi, \xi, \eta). \] (29)
Then, using the notation

\[ \psi(\alpha, \beta, \gamma) := \psi_B(\alpha, \beta, \gamma) := -\alpha^2 B_{\alpha\alpha} - 2\alpha\beta B_{\alpha\beta} - \beta^2 B_{\beta\beta}, \]  

we get the a.e. in \( G^0 \) non-negativity of the following matrix

\[
\begin{bmatrix}
\psi(\alpha, \beta, \gamma), & \frac{1}{2} \alpha B_\alpha \\
\frac{1}{2} \alpha B_\alpha, & -\left(\frac{1}{2} + \gamma^2\right) B_{\gamma\gamma} - \gamma B_\gamma
\end{bmatrix} \geq 0. \]  

(31)

Or,

\[
\begin{bmatrix}
\psi(\alpha, \beta, \gamma), & \frac{1}{2} \alpha B_\alpha \\
\frac{1}{2} \alpha B_\alpha, & -\left(\frac{1}{2} + \gamma^2\right)^{1/2} \left[\left(\frac{1}{2} + \gamma^2\right)^{1/2} B_\gamma\right]_{\gamma}
\end{bmatrix} \geq 0. \]  

(32)

Or, as \( \gamma \ll 1 \)

\[
\begin{bmatrix}
\psi(\alpha, \beta, \gamma), & \frac{1}{2} \alpha B_\alpha \\
\frac{1}{2} \alpha B_\alpha, & -\left[\left(\frac{1}{2} + \gamma^2\right)^{1/2} B_\gamma\right]_{\gamma}
\end{bmatrix} \geq 0. \]  

(33)
26. Mollification of $B$

**Definition**

*Consider a subdomain of $G,*

\[ G_1 := \{(\alpha, \beta, \gamma) \in G : |\gamma| < \frac{1}{2} \alpha, 2 < \beta < Q\}. \]

Denote temporarily $P_t := (t\alpha, t\beta, \gamma), (\alpha, \beta, \gamma) \in G_1, 1/2 \leq t \leq 1$. Then we get for every such $t$ and every point $P_t$ the following inequality for all $(\xi, \eta) \in \mathbb{R}^2$:

\[
\xi^2[\psi(P_t)] + \xi \eta(\alpha t B_\alpha(P_t)) + \eta^2(-[(\frac{1}{2} + \gamma^2)^{1/2} B_\gamma]_\gamma(P_t)) \geq 0.
\]  

(34)
27. Mollified $B$ is $H$

Denote $H(P) = 2 \int_{1/2}^{1} B(P_t) dt$. Notice several simple facts. First of all

$$\alpha H_\alpha = 2 \int_{1/2}^{1} \alpha t B(t \alpha, t \beta, \gamma) dt, \quad \alpha^2 H_{\alpha \alpha} = 2 \int_{1/2}^{1} (\alpha t)^2 B_{\alpha \alpha}(t \alpha, t \beta, \gamma) dt.$$ 

$$\psi_H = -\alpha^2 H_{\alpha \alpha} - 2 \alpha \beta H_{\alpha \beta} - \beta^2 H_{\beta \beta} = 2 \int_{1/2}^{1} \psi_B(t \alpha, t \beta, \gamma) dt.$$ 

Now integrate (34) on the interval $t \in [1/2, 1]$. The previous simple observations allow us now to rewrite this as a pointwise inequality for function $H$ on domain $G_1$ introduced in Definition on slide 26:

$$\xi^2 [\psi_H(P)] + \xi \eta (\alpha H_\alpha(P)) + \eta^2 (-[(1/2 + \gamma^2)^{1/2} B_\gamma]_\gamma(P)) \geq 0.$$ 

(35)
The reader wonders why we are so keen to replace (34) by a virtually the same (35)? The answer is because we can give a very good pointwise estimate on $\psi_H(P)$, $P \in G_1$. Unfortunately we cannot give any pointwise estimate on $\psi(P)$, $P \in G$.

$$R := \sup_{P = (\alpha, \beta, \gamma) \in G} \frac{B(P)}{\alpha}$$  \hspace{1cm} (36)

Our goal formulated in (16) is to prove $R \geq cQ(\log Q)^\epsilon$. We are still not too close, but notice that automatically $B(P) \leq R\alpha$, $P = (\alpha, \beta, \gamma) \in G$.

**Lemma (Main)**

*If $P = (\alpha, \beta, \gamma)$ is such that $|\gamma| \leq \frac{1}{8} \alpha$ and $\beta > 100$ then*

$$\psi_H(P) = 2 \int_{1/2}^{1} \psi(t\alpha, t\beta, \gamma)dt \leq CR(|\gamma| + \frac{\alpha}{\beta}),$$

*where $C$ is an absolute constant.*
Consider function
\[
\varphi(t) := B(t\alpha, t\beta, \gamma)
\] (37)
for a. e. \((\alpha, \beta, \gamma) \in G_1\). It is concave.

Let us first prove that
\[
\int_{1/2}^{1} -\varphi''(t) dt \leq CR(|\gamma| + \frac{\alpha}{\beta}). \quad (38)
\]

This would imply
\[
\int_{1/2}^{1} \psi(t\alpha, t\beta, \gamma) dt \leq CR(|\gamma| + \frac{\alpha}{\beta}),
\]
because we have
\[
\psi(t\alpha, t\beta, \gamma) = -t^2 \varphi''(t).
\]
To prove (38) let us consider an auxiliary function 
\( r(t) := \varphi(1)t - \varphi(t) \). It is defined for 
\( t \in [\max(\frac{\gamma}{\alpha}, \frac{1}{\beta}), 1] \). At 1 it 
vanishes, it is convex, and it attains its maximum on its left 
end-point \( t_0 = \max(\frac{\gamma}{\alpha}, \frac{1}{\beta}) \). The last statement follows from the 
fact that \( \varphi(t)/t \) is increasing: property of \( B \) from slide 13. 
So on \([t_0, 1]\]

\[
r(t) \leq r(t_0) \leq \varphi(1)t_0 \leq R\alpha t_0 \leq R\alpha\left(\frac{\gamma}{\alpha} + \frac{1}{\beta}\right). \tag{39}
\]

As \( \varphi(t)/t \) is increasing, we have \( t\varphi'(t) - \varphi(t) \geq 0 \), and thus 
\( r'(1) \leq 0 \). Let us write down the Taylor formula for convex 
function \( r(t) \) in the integral form, keeping in mind that \( r(1) = 0 \), 
\( r'(1) \leq 0 \): 
\[
 r(t_0) = (t_0 - 1)r'(1) + \int_{t_0}^{1} dt \int_{t}^{1} r''(s)ds.
\]
Fubini’s 
theorem, (39), and \( r'(1) \leq 0 \) imply 
\[
 \int_{t_0}^{1} (s - t_0)r''(s)ds \leq R\alpha\left(\frac{\gamma}{\alpha} + \frac{1}{\beta}\right).
\]

But \( t_0 \leq \frac{1}{8} \) by the assumptions of the lemma. So 
\[
 \int_{1/2}^{1} r''(s)ds \leq \frac{8}{3}R\alpha\left(\frac{\gamma}{\alpha} + \frac{1}{\beta}\right). \text{ Hence, as } r'' = -\varphi'', \text{ we get proof.} 
\]
Let us temporarily take for granted the following inequality, where \( c_1, c_2 \) are absolute positive constants:

\[
\alpha \leq c_2 \frac{\beta}{R} \implies H_\alpha(\alpha, \beta, \gamma) \geq c_1 \beta, \ \beta \in (1, Q/2].
\] (40)
32. Ending the proof

Put

\[ G_3 = \{ P \in G : |\gamma| \leq \frac{1}{1000} \alpha, \beta > 100 \} . \]

By positivity of quadratic form on slide 27, we conclude that for any \( P = (\alpha, \beta, \gamma) \in G_3 \)

\[ [\psi_H] \cdot [-(\frac{1}{2} + \gamma^2)^{1/2} B_\gamma] \gamma \geq \frac{1}{4} \alpha^2 H_\alpha^2. \quad (41) \]

Using the Main Lemma we obtain

\[ \psi_H \leq CR(\gamma + \frac{\alpha}{\beta}) . \]

Now we combine this inequality with the ones on slides 39 and 27 obtain

\[ -(\frac{1}{2} + \gamma^2)^{1/2} B_\gamma] \gamma \geq c_3 \frac{\alpha^2 \beta^2}{R(\frac{\alpha}{\beta} + \gamma)} . \quad (42) \]

Integrate (and use \( \gamma << 1 \))

\[ -H_\gamma \geq c_6 \frac{\alpha^2 \beta^2}{R} \log \left( 1 + \frac{\beta}{\alpha \gamma} \right) . \]
Integrate again:

\[ H(\alpha, \beta, 0) - H(\alpha, \beta, \gamma) \geq c_6 \frac{\alpha^3 \beta}{R} \left[ \left( 1 + \frac{\beta}{\alpha} \gamma \right) \log \left( 1 + \frac{\beta}{\alpha} \gamma \right) - \frac{\beta}{\alpha} \gamma \right] \]

\[ \geq c_7 \frac{\alpha^2 \beta^2}{R} \gamma \log \left( \frac{\beta}{\alpha} \gamma \right), \]  

(43)

the last inequality holds true because \( \frac{\beta}{\alpha} = cR \), and because from now on we will fix \( \alpha, \gamma \) and \( \beta \):

\[ \alpha = c_0 \frac{\beta}{R}, \quad \beta = \frac{Q}{4}, \quad \gamma = c_1 \frac{\beta}{R}, \quad c_1 << c_0. \]  

(44)

We just obtained the following inequality

\[ \frac{\alpha^2 \beta^2}{R} \gamma \log \left( \frac{\beta}{\alpha} \gamma \right) \leq C(H(\alpha, \beta, 0) - H(\alpha, \beta, \gamma)). \]  

(45)
34. Ending the proof

Being even in $\gamma$ on $\gamma \in [-\alpha, \alpha]$ and concave, $H$ automatically decreases for $\gamma \in [0, \alpha]$, concavity and non-negativity of $H$ give $H(\alpha, \beta, \gamma) \geq (1 - \frac{\gamma}{\alpha})H(\alpha, \beta, 0)$. This allows us to estimate the right hand side of (45), and we have

$$\frac{\alpha^2 \beta^2}{R} \gamma \log \left(\frac{\beta}{\alpha} \gamma\right) \leq C(H(\alpha, \beta, 0) - H(\alpha, \beta, \gamma)) \leq C \frac{\gamma}{\alpha} H(\alpha, \beta, 0).$$

Taking into consideration one more time that $H(\alpha, \beta, \gamma) \leq R\alpha$ by the definition of $R$ in (36) and by the construction of $H$, we get

$$\frac{\alpha^2 \beta^2}{R} \gamma \log \left(\frac{\beta}{\alpha} \gamma\right) \leq C(H(\alpha, \beta, 0) - H(\alpha, \beta, \gamma)) \leq CR\gamma.$$

Or, as by our choice of $\alpha, \beta, \gamma$, $\frac{\beta}{\alpha} \gamma \asymp cQ$, we get

$$\frac{Q^4}{R^4} \log \left(\frac{\beta}{\alpha} \gamma\right) \leq C \Rightarrow R \geq cQ(\log Q)^{\frac{1}{4}} \quad (46)$$
35. Improving exponent 1/4 to 1/3

Let us consider the largest \( \tilde{\alpha} \in [\alpha, 1] \), where \( \alpha = \frac{Q}{24R} \) such that the following holds

\[
H(\tilde{\alpha}, \frac{Q}{4}, 0) = \frac{Q}{24}, \quad \text{then} \quad H(\tilde{\alpha}, \frac{Q}{4}, \gamma) \leq \frac{Q}{24}, \quad \gamma \in [0, \tilde{\alpha}]. \tag{47}
\]

Two cases may occur.

Case 1: \( \tilde{\alpha} \geq \frac{Q^{1/2}}{24R^{1/2}} \). Then with these new data, but without any other changes,

\[
c \frac{Q^3}{R^3} \log \left( \frac{cQ}{\tilde{\alpha}} \gamma \right) = c \frac{Q^3}{R^3} \log \left( \frac{cQR^{1/2}}{Q^{1/2}} \cdot \frac{cQ^{1/2}}{R^{1/2}} \right) \leq C. \tag{48}
\]

This implies

\[
R \geq cQ \log^{1/3} Q. \tag{49}
\]
Case 2: $\tilde{\alpha} \leq \frac{Q^{1/2}}{24R^{1/2}}$. At $\alpha_1 := \min\left(\frac{Q}{48R}, \frac{2}{3} \tilde{\alpha}\right)$ we have

$$H(\alpha_1, \frac{Q}{4}, \gamma) \leq \frac{Q}{48}.$$ 

But we saw that $\tilde{\alpha} \geq \frac{Q}{24R}$ by its definition. Hence, $\alpha_1 = \frac{Q}{48\tilde{\alpha}}$. Comparing with (47) we conclude that

$$\tilde{\alpha}H(\alpha_1, \frac{Q}{4}, \gamma) \geq (\tilde{\alpha} - \alpha_1)H(\alpha_1, \frac{Q}{4}, \gamma) \geq$$

$$H(\tilde{\alpha}, \frac{Q}{4}, \gamma) - H(\alpha_1, \frac{Q}{4}, \gamma) \geq (1 - \frac{\gamma}{\tilde{\alpha}})H(\tilde{\alpha}, \frac{Q}{4}, 0) - \frac{Q}{48} \geq$$

$$(1 - \frac{\gamma}{\tilde{\alpha}})H(\tilde{\alpha}, \frac{Q}{4}, 0) - \frac{Q}{48} \geq (1 - \frac{\gamma}{\tilde{\alpha}})\frac{Q}{24} - \frac{Q}{48} = \frac{Q}{144},$$

if $\gamma \in [0, \frac{2}{3} \alpha_1]$.

Using $\tilde{\alpha} \leq \frac{Q^{1/2}}{24R^{1/2}}$, we get the improved estimate on the derivative:

$$\forall \gamma \in [0, \frac{2}{3} \alpha_1] \quad H(\alpha_1, \frac{Q}{4}, \gamma) \geq cQ^{1/2}R^{1/2} \quad (50)$$

$$\Rightarrow c \frac{Q^2}{R^2} \frac{QR}{R} \log \left(\frac{cQ}{\alpha_1 \gamma}\right) \leq CR, \Rightarrow R \geq cQ \log^{1/3} Q.$$
What follows is a joint work with Paata Ivanisvili.

**Theorem**

If a real valued function $M(x, y)$ is such that $M(x, \sqrt{y}) \in C^2(\Omega \times \mathbb{R}_+)$ and it satisfies the differential inequalities

$$
\begin{bmatrix}
M_{xx} + \frac{M_y}{y} & M_{xy} \\
M_{xy} & M_{yy}
\end{bmatrix} \leq 0 \quad \text{and} \quad M_y \leq 0,
$$

then for any $f \in C_0^\infty(\mathbb{R}^n; \Omega)$ we have

$$
\int_{\mathbb{R}^n} M(f, \|\nabla f\|) d\gamma \leq M \left( \int_{\mathbb{R}^n} f d\gamma, 0 \right).
$$
37. Log-Sobolev inequality

\[ M(x, y) = x \ln x - \frac{y^2}{2x}, \quad x > 0 \text{ and } y \geq 0. \] (53)

Notice that \( M(x, y) \) satisfies (51). Indeed, \( M_y = -\frac{y}{x} \leq 0 \) and

\[
\begin{bmatrix}
M_{xx} + \frac{M_y}{y} & M_{xy} \\
M_{xy} & M_{yy}
\end{bmatrix} = \begin{bmatrix}
-\frac{y^2}{x^3} & \frac{y}{x^2} \\
\frac{y}{x^2} & -\frac{1}{x}
\end{bmatrix} \leq 0.
\] (54)

Log-Sobolev inequality of Gross states that

\[
\int_{\mathbb{R}^n} |f|^2 \ln |f|^2 \, d\gamma - \left( \int_{\mathbb{R}^n} |f|^2 \, d\gamma \right) \ln \left( \int_{\mathbb{R}^n} |f|^2 \, d\gamma \right) \leq 2 \int_{\mathbb{R}^n} \| \nabla f \|^2 \, d\gamma
\] (55)

whenever the right hand side of (55) is well-defined and finite for complex-valued \( f \).
38. Beckner–Sobolev and spectral gap inequality

Beckner:

For \( f \in L^2(d\gamma) \) and \( 1 \leq p \leq 2 \) we have

\[
\int |f|^2 d\gamma - \left( \int |f|^p d\gamma \right)^{2/p} \leq (2 - p) \int_{\mathbb{R}^n} \|\nabla f\|^2 d\gamma \tag{56}
\]

For \( p = 1 \) this is \( \int |f|^2 d\gamma - \left( \int |f| d\gamma \right)^2 \leq \int_{\mathbb{R}^n} \|\nabla f\|^2 d\gamma \). This shows that the spectral gap i.e. the first nontrivial eigenvalue of the self-adjoint positive operator \( L = -\Delta + x \cdot \nabla \) in \( L^2(\mathbb{R}^n, d\gamma) \) is bounded from below by 1.

\[
M(x, y) = x^{\frac{2}{p}} - \frac{2-p}{p^2} x^{\frac{2}{p} - 2} y^2 \quad \text{where} \quad x, y \geq 0 \quad 1 \leq p \leq 2. \]

Notice that \( M_y = -\frac{2(2-p)}{p^2} x^{\frac{2}{p} - 2} y \leq 0 \) and

\[
\begin{bmatrix}
M_{xx} + \frac{M_y}{y} & M_{xy} \\
M_{xy} & M_{yy}
\end{bmatrix} =
\begin{bmatrix}
\frac{2(2-p)(1-p)(2-3p)x^{\frac{2-4}{p}} y^2}{p^4} & \frac{4(2-p)(1-p)x^{\frac{2-3}{p}} y}{p^3} \\
\frac{4(2-p)(1-p)x^{\frac{2-3}{p}} y}{p^3} & \frac{4(2-p)x^{\frac{2-2}{p}}}{p^2}
\end{bmatrix} \leq C \tag{57}
\]
For a Lipschitz function $f: \mathbb{R}^n \to [0, 1]$, we have

$$I \left( \int_{\mathbb{R}^n} f d\gamma \right) \leq \int_{\mathbb{R}^n} \sqrt{l^2(f) + \|\nabla f\|^2} d\gamma,$$

(58)

where $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-x^2/2} dx$, and $I(x) := \Phi'(\Phi^{-1}(x))$.

Testing (58) for $f(x) = 1_A$ where $A$ is a Borel subset of $\mathbb{R}^n$ one obtains Gaussian isoperimetry: for any Borel measurable set $A \subset \mathbb{R}^n$

$$\gamma^+(A) \geq \Phi'(\Phi^{-1}(\gamma(A))),$$

(59)

where $\gamma^+(A) := \liminf_{\varepsilon \to 0} \frac{\gamma(A_{\varepsilon}) - \gamma(A)}{\varepsilon}$ denotes Gaussian perimeter of $A$, here $A_{\varepsilon} = \{ x \in \mathbb{R}^n : \text{dist}_{\mathbb{R}^n}(A, x) < \varepsilon \}$. 
40. Bobkov’s inequality: Gaussian isoperimetry

\[ M(x, y) = -\sqrt{I^2(x) + y^2} \quad \text{where} \quad x \in [0, 1], \quad y \geq 0. \quad (60) \]

Then \( M_y = \frac{-y}{\sqrt{I^2(x) + y^2}} \leq 0 \) and

\[
\begin{bmatrix}
M_{xx} + \frac{M_y}{y} & M_{xy} \\
M_{xy} & M_{yy}
\end{bmatrix} = \begin{bmatrix}
-\frac{(I'(x))^2y^2}{(I^2(x)+y^2)^{3/2}} + \frac{I(x)I''(x)+1}{\sqrt{I^2(x)+y^2}} & y \frac{I(x)I'(x)}{(I^2(x)+y^2)^{3/2}} \\
y \frac{I(x)I'(x)}{(I^2(x)+y^2)^{3/2}} & -\frac{I^2(x)}{(I^2(x)+y^2)^{3/2}}
\end{bmatrix}
\]

(61)

Notice that \( I''(x)I(x) = -1 \), therefore (61) is negative semidefinite.
In general finding $M(x, y)$ will be based purely on solving PDEs. First notice that in log-Sobolev (55) and in Bobkov’s inequality (58) determinant of the matrices (54) and (61) are zero. In Beckner–Sobolev inequality (56) determinant of (57) is zero if and only if $p = 1, 2$. We will seek $M(x, y)$ among those functions which in addition with (51) also satisfy Monge–Ampère equation with a drift:

$$\det \begin{bmatrix} M_{xx} + \frac{M_y}{y} & M_{xy} \\ M_{xy} & M_{yy} \end{bmatrix} = M_{xx} M_{yy} - M_{xy}^2 + \frac{M_y M_{yy}}{y} = 0 \quad (62)$$

for $(x, y) \in \Omega \times \mathbb{R}_+$. 
42. Reduction to the exterior differential systems and backwards heat equation

Let us make the following observation: consider

\[(x, y, p, q) = (x, y, M_x(x, y), M_y(x, y))\]

in \(xypq\)-space. This is a surface \(\Sigma\) in 4-space on which \(\gamma = dx \wedge dy\) is nonvanishing but to which the two 2-forms

\[\gamma_1 = dp \wedge dx + dq \wedge dy\]

and

\[\gamma_2 = (ydp + qdx) \wedge dq\]

pull back to be zero. Consider a simply connected surface \(\Sigma\) in \(xypq\)-space (with \(y > 0\)) on which \(\gamma\) is nonvanishing but to which \(\gamma_1\) and \(\gamma_2\) pullback to be zero. The 1-form \(pdx + qdy\) pull back to \(\Sigma\) to be closed (since \(\gamma_1\) vanishes on \(\Sigma\)) and hence exact, and therefore there exists a function \(m : \Sigma \to \mathbb{R}\) such that

\[dm = pdx + qdy\]

on \(\Sigma\). We then have (at least locally),

\(m = M(x, y)\) on \(\Sigma\) and, by its definition, we have \(p = M_x(x, y)\) and \(q = M_y(x, y)\) on the surface. \(\gamma_2\) vanishes when pulled back to \(\Sigma\) implies that \(M(x, y)\) satisfies the desired equation (62).
Thus, we have encoded the given PDE as an exterior differential system on $\mathbb{R}^4$. Note, that we can make a change of variables on the open set where $q < 0$: Set $y = qr$ and let $t = \frac{1}{2} q^2$. Then, using these new coordinates on this domain, we have

$$\gamma_1 = dp \wedge dx + dt \wedge dr \quad \text{and} \quad \gamma_2 = (rdp + dx) \wedge dt.$$

Now, when we take an integral surface $\Sigma$ on these 2-forms on which $dp \wedge dt$ is not vanishing, it can be written locally as a graph of the form

$$(p, t, x, r) = (p, t, u_p(p, t), u_t(p, t))$$

(since $\Sigma$ is an integral of $\gamma_1$), where $u(p, t)$ satisfies $u_t + u_{pp} = 0$ (since on $\Sigma$ $0 = \gamma_2 = u_t dp \wedge dt + du_p \wedge dt = (u_t + u_{pp}) dp \wedge dt$). Thus, “generically” our PDE is equivalent to the backwards heat equation, up to a change of variables.
Thus the function \( M(x, y) \) can be parametrized as follows:

\[
x = u_p \left( p, \frac{1}{2} q^2 \right) ; \quad y = qu_t \left( p, \frac{1}{2} q^2 \right) ;
\]

\[
M(x, y) = pu_p \left( p, \frac{1}{2} q^2 \right) + q^2 u_t \left( p, \frac{1}{2} q^2 \right) - u \left( p, \frac{1}{2} q^2 \right) ,
\]

where

\[
u_t + u_{pp} = 0 .
\]

\( M(x, \sqrt{y}) \in C^2(\Omega \times \mathbb{R}_+) \) therefore \( M_y(x, 0) = 0 \). By choosing \( y = 0 \) in (63), we have \( q = 0 \), and we obtain the boundary condition:

\[
M(x, 0) = M_x(x, 0) \cdot x - M_y(x, 0) \cdot y|_{y=0} = u(M_x(x, 0), 0) .
\]

Or, if to denote boundary function \( M(x, 0) \) by \( f(x) \), then \( u \) has initial conditions \( (t = 0, \text{ that is } q^2 = (M_y(x, 0))^2 = 0) \):

\[
u(f'(x), 0) = xf'(x) - f(x) , \quad f(x) = M(x, 0) .
\]

Non-negativity of matrix also implies one more condition.
45. Applications: how to find Bellman log-Sobolev function

In this case inequality (55) shows us sharp lower bounds of the expression $(\int g d\gamma) \ln (\int g d\gamma)$. Therefore, we should take $M(x, 0) = x \ln x$. Boundary condition then can be rewritten as $u(\ln x + 1, 0) = x$ or $u(p, 0) = e^{p-1}$ for all $p \in \mathbb{R}$. If we set $D = \frac{\partial^2}{\partial p^2}$ then

$$u(p, t) = e^{-tD} e^{p-1} = \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} e^{p-1} = e^{p-t-1} \quad \text{for all } t \geq 0.$$

Clearly $u(p, t)$ satisfies (64) because $\det(\text{Hess } u) = 0$. Notice that we have $u_t < 0$, 

$$\begin{cases}
  x = e^{p - \frac{q^2}{2} - 1}, \\
  y = -q e^{p - \frac{q^2}{2} - 1};
\end{cases} \quad \text{then} \quad \begin{cases}
  q = -\frac{y}{x}; \\
  p = \ln x + \frac{y^2}{2x^2} + 1.
\end{cases}$$

Therefore we obtain

$$M(x, y) = xp + qy - u \left( p, \frac{1}{2} q^2 \right) = x \ln x + \frac{y^2}{2x} + x - \frac{y^2}{x} - x = x \ln x - \frac{y^2}{2x}.$$
In this case we are interested for the sharp lower bounds of the expression $-I(\int f d\gamma)$ in terms of $\int M(f, \|\nabla f\|)d\gamma$. We have $M(x, 0) = -I(x)$. Boundary condition takes the form

$$u(p, 0) = p\Phi(p) + \Phi'(p) \quad \text{for all} \quad p \in \mathbb{R}. \quad (65)$$

In fact, $M_x(x, 0) = -l'(x)$ and $-l'(x) = \Phi^{-1}(x)$:

$$l'(x) = \left[ e^{-\frac{[\Phi^{-1}]^2}{2}} \right]' \quad \text{and} \quad (\Phi^{-1})' = e^{\frac{[\Phi^{-1}]^2}{2}}. \quad \text{Now we will try to find}$$

the usual heat extension of $u(p, 0)$ (call it $\tilde{u}(p, t)$) which satisfies $\tilde{u}_{pp} = \tilde{u}_t$, and then we try to consider the formal candidate $u(p, t) := \tilde{u}(p, -t)$. The heat extension of $\Phi'(p) = \frac{1}{\sqrt{2\pi}} e^{-p^2/2}$ is

$$\frac{1}{\sqrt{2\pi}} \frac{p^2}{\sqrt{1+2t}} e^{-\frac{p^2}{2(1+2t)}}. \quad \text{Heat extension of} \quad \Phi(p) \quad \text{is} \quad \Phi \left( \frac{p}{\sqrt{1+2t}} \right). \quad \text{Indeed,}$$

the heat extension of the function $1_{(-\infty, 0]}(p)$ at time $t = 1/2$ is $\Phi(p)$. By the semigroup property the heat extension of $\Phi(p)$ at time $t$ will be the heat extension of $1_{(-\infty, 0]}(p)$ at time $1/2 + t$ which equals to $\Phi \left( \frac{p}{\sqrt{1+2t}} \right)$. 

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End point estimates and Monge–Ampère
Therefore, the heat extension of $p\Phi(p)$ can be found as follows:

$$\frac{2t}{\sqrt{2\pi\sqrt{1+2t}}} e^{-\frac{p^2}{2(1+2t)}} + p\Phi \left( \frac{p}{\sqrt{1+2t}} \right).$$

Thus we obtain that

$$\tilde{u}(p, t) = \sqrt{1+2t} \Phi' \left( \frac{p}{\sqrt{1+2t}} \right) + p\Phi \left( \frac{p}{\sqrt{1+2t}} \right).$$

This expression is well defined even for $t \in (0, -1/2)$. Therefore if we set

$$u(p, t) = \tilde{u}(p, -t) = \sqrt{1-2t} \Phi' \left( \frac{p}{\sqrt{1-2t}} \right) + p\Phi \left( \frac{p}{\sqrt{1-2t}} \right) \quad \text{for} \quad p \in \mathbb{R}.$$
Direct computations show that $u(p, t)$ satisfies $u_t + u_{pp} = 0$, the boundary condition (65) and (64) because

$$\det(\text{Hess } u) = -\left( \frac{\Phi'(p/(\sqrt{1-2t}))}{1-2t} \right)^2 < 0.$$ We have $u_t = -\frac{\Phi'(p/(\sqrt{1-2t}))}{\sqrt{1-2t}} < 0$ and $u_p = \Phi\left(\frac{p}{\sqrt{1-2t}}\right)$. Therefore,

$$\begin{cases} x = \Phi\left(\frac{p}{\sqrt{1-q^2}}\right); \\
y = qr = qu_t = \frac{-q}{\sqrt{1-q^2}} \Phi'(\frac{p}{\sqrt{1-q^2}}); \end{cases} \text{ then } \begin{cases} \Phi^{-1}(x) = \frac{p}{\sqrt{1-q^2}}; \\
y = \frac{-q}{\sqrt{1-q^2}} \Phi'(\Phi^{-1}(x)). \end{cases}$$

From the last equalities we obtain $M_y = q = \frac{-y}{\sqrt{l^2(x)+y^2}}$ and $M_x = p = \frac{l(x)\Phi^{-1}(x)}{\sqrt{l^2(x)+y^2}}$ where we remind that $l(x) = \Phi'(\Phi^{-1}(x))$. Then it is clear that

$$M(x, y) = -\sqrt{l^2(x) + y^2}.$$