Compensated Compactness
Interpolatory Estimates
Riesz Transforms, Wavelet- and Haar Projections

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Helsinki 2016
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Primary Sources

- J. Lee, S. Müller and PFXM, Compensated Compactness, separately convex Functions and interpolatory estimates between Riesz transforms and Haar projections. 
Primary Sources


Fourier- and Riesz Transforms

\[ \mathcal{F}(u) (\tau) = \frac{1}{p} (2\pi)^{-n/2} \int_{\mathbb{R}^n} u(x) e^{i x \cdot \tau} \, dx \]

The Riesz transformation as Fourier multiplier

\[ \mathcal{F}(R_j(u)) (\tau) = \frac{1}{\mathcal{F}(u) (\tau)} \tau_j |\tau| \text{ with } 1 \leq j \leq n, \tau = (\tau_1, \ldots, \tau_n). \]

\( R_j \) is \( L^p(\mathbb{R}^n) \) bounded

\[ k_{R_j} \leq C p^{-2/(p-1)}, 1 < p < 1. \]
Fourier- and Riesz Transforms

Fourier Transformation

\[ \mathcal{F}(u)(\xi) = \frac{1}{\sqrt{(2\pi)^n}} \int_{\mathbb{R}^n} u(x) e^{-ix\cdot \xi} \, dx \]
Fourier- and Riesz Transforms

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The Riesz transformation as Fourier multiplier

\[ \mathcal{F}(R_j(u))(\xi) = -\sqrt{-1} \frac{\xi_i}{|\xi|} \mathcal{F}(u)(\xi) \quad \text{with} \quad 1 \leq i \leq n, \quad \xi = (\xi_1, \ldots, \xi_n). \]
Fourier- and Riesz Transforms

Fourier Transformation

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\( R_j \) is \( L^p(\mathbb{R}^n) \)-bounded

\[ \| R_j \|_p \leq C p^2 / (p - 1), \quad 1 < p < \infty. \]
Lower Semi-continuity

**Theorem (J. Lee, S. Müller, P.F.X.M.)**

Let $f : \mathbb{R}^n \to \mathbb{R}^+$ be separately convex and polynomially bounded.
**The Result**

**Lower Semi-continuity**

**Theorem (J. Lee, S. Müller, P.F.X.M.)**

Let $f : \mathbb{R}^n \to \mathbb{R}^+$ be separately convex and polynomially bounded. If $(v_r : \mathbb{R}^n \to \mathbb{R}^n)$ is weakly convergent

$v_r \rightharpoonup v$ weakly in $L^p(\mathbb{R}^n, \mathbb{R}^n)$,

then we have lower semi-continuity:

$$
\int_{\mathbb{R}^n} f(v(x)) \, dx \leq \liminf_{r \to 0} \int_{\mathbb{R}^n} f(v_r(x)) \, dx,
$$

provided that $\sum_{i \neq j \neq k} R_i(v_j(r)) k \to 0$.
**Lower Semi-continuity**

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\]

then we have lower semi-continuity: \( \forall \varphi \geq 0, \)

\[
\int_{\mathbb{R}^n} f(v(x))\varphi(x)dx \leq \liminf_{r \to \infty} \int_{\mathbb{R}^n} f(v_r(x))\varphi(x)dx,
\]
The Result

Lower Semi-continuity

**Theorem (J. Lee, S. Müller, P.F.X.M.)**

Let \( f : \mathbb{R}^n \to \mathbb{R}^+ \) be **separately convex** and polynomially bounded. If \( (v_r : \mathbb{R}^n \to \mathbb{R}^n) \) is weakly convergent

\[ v_r \rightharpoonup v \text{ weakly in } L^p(\mathbb{R}^n, \mathbb{R}^n), \]

then we have **lower semi-continuity**: \( \forall \varphi \geq 0, \)

\[ \int_{\mathbb{R}^n} f(v(x)) \varphi(x) \, dx \leq \liminf_{r \to \infty} \int_{\mathbb{R}^n} f(v_r(x)) \varphi(x) \, dx, \]

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\]

provided that

\[
\forall i \neq j \quad \|R_i(v_r^{(j)})\|_{L^p(\mathbb{R}^n)} \to 0.
\]
Compensated Compactness

Compensated Compactness obtains

- weak continuity results for non linear functionals,
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- weak continuity results for non linear functionals,
- and weak lower semicontinuity of non convex Lagrangians.
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This is possible only when weakly converging testing functions satisfy additional constrains.
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First order systems of Electrostatics

\[ E_k(x) = -\operatorname{grad}V_k(x), \quad D_k(x) = \epsilon_k(x)E_k(x), \quad \operatorname{div}D_k(x) = \rho_k(x) \]
Departure

**Theorem (Murat Tartar):**

Weak convergence of the vector fields

\[ E_k \rightharpoonup E, \quad D_k \rightharpoonup D, \quad \text{in} \quad L^2_{\text{lok}}(\mathbb{R}^3) \]
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and weak convergence of the charge densities

\[ \rho_k \rightharpoonup \rho \quad \text{in} \quad L^2_{\text{loc}}(\mathbb{R}^3), \]
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\[ \rho_k \rightharpoonup \rho \quad \text{in} \quad L^2_{\text{loc}}(\mathbb{R}^3), \]

implies weak convergence of the products (the energy densities)

\[ D_k \cdot E_k \rightharpoonup D \cdot E \quad \text{in} \quad L^1_{\text{loc}}(\mathbb{R}^3). \]


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**Compare:**

\[ \sin(kx) \rightharpoonup 0, \quad \sin(kx) \cdot \sin(kx) \rightharpoonup \frac{1}{2} > 0. \]
What is special about

\[ E_k(x) = -\nabla V_k(x), \quad D_k(x) = \varepsilon_k(x) E_k(x), \quad \text{div} D_k(x) = \rho_k(x)? \]
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Gradients are curl-free

$$\operatorname{curl} E = (\partial_i E^j - \partial_j E^i)_{i,j=1}^3 = 0.$$
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Compactness of Sobolev embedding:

\[ \|f\|_{W^{-1,p}(\mathbb{R}^n)} = \|\mathcal{F}^{-1}\{\mathcal{F} f(\xi)(1 + |\xi|^2)^{-1/2}\}\|_{L^p(\mathbb{R}^n)}. \]

\( \text{Id} : L^2 \to W^{-1,2} \) maps weak convergence to norm convergence, hence: \( \text{div} D_k \) is \( W^{-1,2} \) convergent.
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Compactness of Sobolev embedding:

\[ \|f\|_{W^{-1,p}(\mathbb{R}^n)} = \|F^{-1}\{Ff(\xi)(1 + |\xi|^2)^{-1/2}\}\|_{L^p(\mathbb{R}^n)} : \]

Id : \(L^2 \rightarrow W^{-1,2}\) maps weak convergence to norm convergence, hence:

\(\text{div} D_k\) is \(W^{-1,2}\) convergent.

Orthogonality

\(\text{curl} X = 0, \quad \text{div} Y = 0\) implies \(\int X \cdot Y dx = 0.\)
Hodge Decomposition

The Riesz transform

\[ R_j(f)(x) = i \mathcal{F}^{-1}(y_j/|y| \cdot \mathcal{F}(f))(x) \]
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Projections onto divergence free and curl free vector fields.

If \( w = (w^1, \ldots, w^n) : \mathbb{R}^n \to \mathbb{R}^n \) and

\[ P(w) = (R_i \otimes R_j)(w) = (R_i(\sum_{j=1}^{n} R_j w^{(j)}))_{i=1}^{n} \]

then
Hodge Decomposition

The Riesz transform
\[ R_j(f)(x) = iF^{-1}\left(y_j/|y| \cdot F(f)(x)\right) \]

Projections onto divergence free and curl free vector fields.
If \( w = (w^1, \ldots, w^n) : \mathbb{R}^n \to \mathbb{R}^n \) and
\[ P(w) = (R_i \otimes R_j)(w) = (R_i(\sum_{j=1}^{n} R_j w^{(j)}))_{i=1}^{n} \]
then
\[ \text{curl}(Pw) = 0, \quad P^2 = P, \quad \text{div}(w - P(w)) = 0. \]
The div-curl Lemma (Murat-Tartar):

For sequences \((v_k : \mathbb{R}^n \to \mathbb{R}^n)\) and \((w_k : \mathbb{R}^n \to \mathbb{R}^n)\) satisfying

\[
\begin{align*}
w_k &\rightharpoonup 0 \text{ weakly in } L^p_{\text{lok}}(\mathbb{R}^n, \mathbb{R}^n), \\
v_k &\rightharpoonup 0 \text{ weakly in } L^q_{\text{lok}}(\mathbb{R}^n, \mathbb{R}^n),
\end{align*}
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and

\[ \|\text{curl}w_k\|_{W^{(-1,p)}} \to 0, \quad \|\text{div}v_k\|_{W^{(-1,q)}} \to 0, \]

there exist decompositions
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and \(\|\text{curl}w_k\|_{W(-1,p)} \to 0, \quad \|\text{div}v_k\|_{W(-1,q)} \to 0,\)

there exist decompositions

\[ w_k = z_k + r_k, \quad v_k = z'_k + r'_k, \quad \text{so that} \]

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\[ w_k = z_k + r_k, \quad v_k = z_k' + r_k', \quad \text{so that} \]

\[ \text{curl} z_k = 0, \quad \text{div} z_k' = 0, \quad \|r_k\|_{L^p} \to 0, \quad \text{and} \quad \|r_k'\|_{L^p} \to 0. \]
The div-curl Lemma (Murat-Tartar):

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\text{curl} z_k = 0, \quad \text{div} z_k' = 0, \quad \|r_k\|_{L^p} \to 0, \quad \text{and} \quad \|r_k'\|_{L^p} \to 0.
\]
By orthogonality and
\[
(z_k + r_k)(z_k' + r_k') = (z_k z_k' + z_k r_k' + z_k' r_k + r_k r_k'),
\]
\[
\int v_k \cdot w_k \to 0.
\]
Proof of Div-curl Lemma

\[
\frac{\dot{\zeta}_i}{|\zeta|} = \frac{\dot{\zeta}_i}{1 + |\zeta|} + \frac{\dot{\zeta}_i}{|\zeta|(1 + |\zeta|)}
\]
implies

\[
||\nabla_j \mathbf{w}||_p \leq ||\partial_j \mathbf{w}||_{W^{-1,p}} + ||T(\mathbf{w})||_p.
\]

where $T$ is a compact operator.
Proof of Div-curl Lemma

\[ \frac{\zeta_i}{|\zeta|} = \frac{\zeta_i}{1 + |\zeta|} + \frac{\zeta_i}{|\zeta|(1 + |\zeta|)} \]

implies \[ ||R_j w||_p \leq ||\partial_j w||_{W^{-1,p}} + ||T(w)||_p. \]

where \( T \) is a compact operator. Hence with

\[
( \text{Id} - P)w = ((\sum_{j=1}^{n} R_j(R_j w^{(i)} - R_i w^{(j)})))_{i=1}^{n},
\]

\[ ||( \text{Id} - P)w||_p \leq ||\text{curl}w||_{W^{-1,p}} + ||\text{compact}(w)||_p. \]
Proof of Div-curl Lemma

\[ \frac{\zeta_i}{|\zeta|} = \frac{\zeta_i}{1 + |\zeta|} + \frac{\zeta_i}{|\zeta|(1 + |\zeta|)} \]

implies

\[ \| R_j w \|_p \leq \| \partial_j w \|_{W^{-1,p}} + \| T(w) \|_p. \]

where \( T \) is a compact operator. Hence with

\[ (\text{Id} - P)w = (\sum_{j=1}^{n} R_j(R_j w^{(i)} - R_i w^{(j)}))_{i=1}^{n}, \]

\[ \|(\text{Id} - P)w\|_p \leq \|\text{curl}w\|_{W^{-1,p}} + \|\text{compact}(w)\|_p. \]

If \( w_k \rightharpoonup w \) weakly in \( L^p \) and \( \|\text{curl}w_k\|_{W^{-1,p}} \to 0 \) then decompose

\[ w_k = P w_k + (\text{Id} - P)w_k, \]

to obtain

\[ \text{curl}Pw_k = 0, \quad \|(\text{Id} - P)w_k\|_p \to 0. \]

Same for \( Q = (\text{Id} - P) \) and divergence.
Semi continuity and convexity.

Let $1 < p < \infty$, $f : \mathbb{R}^n \to \mathbb{R}^+$, $0 < f(x) \leq C(1 + |x|^p)$. 
Semi continuity and convexity.

Let $1 < p < \infty$, $f : \mathbb{R}^n \to \mathbb{R}^+$, $0 < f(x) \leq C(1 + |x|^p)$.

**A) Fatou on lower semi continuity**

If $f$ is continuous and $w_j \to w$ in $L^p$ norm convergent, then

$$\int_{[0,1]^n} f(w(x))dx \leq \liminf_{j \to \infty} \int_{[0,1]^n} f(w_j(x))dx.$$
Semi continuity and convexity.

Let $1 < p < \infty$, $f : \mathbb{R}^n \to \mathbb{R}^+$, $0 < f(x) \leq C(1 + |x|^p)$.

A) Fatou on lower semi continuity
If $f$ is continuous and $w_j \to w$ in $L^p$ norm convergent, then

$$\int_{[0,1]^n} f(w(x)) \, dx \leq \liminf_{j \to \infty} \int_{[0,1]^n} f(w_j(x)) \, dx.$$ 

B) Hahn-Banach on lower semi continuity
If $f$ is convex and $w_j \rightharpoonup w$ in $L^p$ weakly convergent then again

$$\int_{[0,1]^n} f(w(x)) \, dx \leq \liminf_{j \to \infty} \int_{[0,1]^n} f(w_j(x)) \, dx,$$

and conversely.
Weak lower semi continuity implies convexity

Let \( h : \mathbb{R}^n \rightarrow \mathbb{R}^n \) be bounded and \([0, 1]^n\) periodic. We prove

\[
f(\int_{[0,1]^n} h(x)dx) \leq \int_{[0,1]^n} f(h(x))dx.
\]
Weak lower semi continuity implies convexity

Let \( h : \mathbb{R}^n \rightarrow \mathbb{R}^n \) be bounded and \([0,1]^n\) periodic. We prove

\[
 f\left( \int_{[0,1]^n} h(x) \, dx \right) \leq \int_{[0,1]^n} f(h(x)) \, dx.
\]

Form highly oscillatory and hence weakly convergent, \( w_j(x) = h(jx) \),

\[
 w_j \rightarrow \int_{[0,1]^n} h(x) \, dx.
\]
Weak lower semi continuity implies convexity

Let $h : \mathbb{R}^n \to \mathbb{R}^n$ be bounded and $[0,1]^n$ periodic. We prove

$$f\left(\int_{[0,1]^n} h(x) \, dx\right) \leq \int_{[0,1]^n} f(h(x)) \, dx.$$ 

Form highly oscillatory and hence weakly convergent, $w_j(x) = h(jx)$,

$$w_j \rightarrow \int_{[0,1]^n} h(x) \, dx.$$ 

Weak lower semi continuity implies Jensen’s Inequality

$$f\left(\int_{[0,1]^n} h(x) \, dx\right) \leq \liminf_{j \to \infty} \int_{[0,1]^n} f(w_j(x)) \, dx = \int_{[0,1]^n} f(h(x)) \, dx.$$
Interpretation

Comparison of A and B:

**Extending** the class of admissible testing sequences from norm-converging to weakly converging **is compensated by restricting** the class of admissible integrands $f$ from continuous to convex.
Interpretation

Comparison of A and B:

Extending the class of admissible testing sequences from norm-converging to weakly converging is compensated by restricting the class of admissible integrands $f$ from continuous to convex.

Emphasize

The weakly convergent sequence satisfies Jensen’s inequality with respect to $f$. 
Quasi-Convexity

Gradients are curl-free

\[ w : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}. \]

\[ \text{curl} w = (\partial_i w^{(m,j)} - \partial_j w^{(m,i)})_{i,j=1}^n; \quad m \leq n. \]
Quasi-Convexity

Gradients are curl-free

\[ w : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n} \]

\[ \text{curl} w = (\partial_i w^{(m,i)} - \partial_j w^{(m,i)})_{i,j=1}^n, \quad m \leq n. \]

If \( v : \mathbb{R}^n \rightarrow \mathbb{R}^n \) and \( \partial_i v^{(m)} = w^{(m,i)} \) then \( \text{curl} w = 0. \)
Quasi-Convexity

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\[ w : \mathbb{R}^n \to \mathbb{R}^{n \times n}. \]

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\text{curl} w = (\partial_i w^{(m,j)} - \partial_j w^{(m,i)})_{i,j=1}^n, \quad m \leq n.
\]

If \( v : \mathbb{R}^n \to \mathbb{R}^n \) and \( \partial_i v^{(m)} = w^{(m,i)} \) then \( \text{curl} w = 0. \)

Jensen’s inequality for gradients = quasi convex.

If \( L : \mathbb{R}^{n \times n} \to \mathbb{R}^+ \), \( L(x) \leq (1 + |x|^p) \) satisfies Jensen’s inequality for gradients,

\[
\int_{[0,1]^n} L(a + w) \geq L(a), \quad \int_{[0,1]^n} w = 0, \quad \text{curl} w = 0.
\]

then \( L : \mathbb{R}^{n \times n} \to \mathbb{R}^+ \) is called quasi convex.
Morrey’s Theorem

Assume quasi convexity, $L: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$, and $Z[0,1]^n \subseteq L(a+w)$ and $Z[0,1]^n w = 0$, $\nabla w = 0$.

If $(w_r v)$ weakly in $L^p(\mathbb{R}^n, \mathbb{R}^n)$, $\nabla (w_r v) = 0$, then $Z L(w) dx \leq \lim \inf Z L(w_k) dx$, and conversely.
Morrey’s Theorem

Assume quasi convexity, $L : \mathbb{R}^{n \times n} \to \mathbb{R}^+$, $L(x) \leq (1 + |x|^p)$ and

$$\int_{[0,1]^n} L(a + w) \geq L(a), \quad \int_{[0,1]^n} w = 0, \quad \text{curl} w = 0.$$
**Morrey’s Theorem**

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\]

If

\[
\begin{align*}
&\{ w_r \rightharpoonup v \text{ weakly in } L^p(\mathbb{R}^n, \mathbb{R}^{n \times n}), \\
&\text{curl}(w_r) = 0,
\end{align*}
\]

then

\[
\int_{[0,1]^n} L(a + w) \geq L(a), \quad \int_{[0,1]^n} w = 0, \quad \text{curl} w = 0.
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Morrey’s Theorem

Assume quasi convexity, $L : \mathbb{R}^{n \times n} \to \mathbb{R}^+$, $L(x) \leq (1 + |x|^p)$ and

$$\int_{[0,1]^n} L(a + w) \geq L(a), \quad \int_{[0,1]^n} w = 0, \quad \text{curl} w = 0.$$  

If

$$\begin{cases}
w_r \rightharpoonup v \quad \text{weakly in} \quad L^p(\mathbb{R}^n, \mathbb{R}^{n \times n}), \\
\text{curl}(w_r) = 0,
\end{cases}$$

then

$$\int L(w) dx \leq \lim \inf \int L(w_k) dx,$$

and conversely.
Morrey's Theorem extended by Murat and Tartar.

Assume quasi convexity, \( L : \mathbb{R}^{n \times n} \to \mathbb{R}^+ \), \( L(x) \leq (1 + |x|^p) \) and

\[
\int_{[0,1]^n} L(a + w) \geq L(a), \quad \int_{[0,1]^n} w = 0, \quad \text{curl}w = 0.
\]

If

\[
\begin{align*}
\{ w_r \rightharpoonup v \quad \text{weakly in} \quad & L^p(\mathbb{R}^n, \mathbb{R}^{n \times n}), \\
\text{curl}(w_r) \quad & \text{pre-compact in} \quad W^{-1,p}(\mathbb{R}^n, \mathbb{R}^{n \times n}),
\end{align*}
\]

then

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\int L(w) dx \leq \liminf \int L(w_k) dx.
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Morrey’s Theorem extended by Murat and Tartar.

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If

\[
\begin{cases}
    w_r \to v \quad \text{weakly in} \quad L^p(\mathbb{R}^n, \mathbb{R}^{n \times n}), \\
    \text{curl}(w_r) \quad \text{pre-compact in} \quad W^{-1,p}(\mathbb{R}^n, \mathbb{R}^{n \times n}),
\end{cases}
\]

then

\[
\int L(w)dx \leq \liminf \int L(w_k)dx.
\]

Note: \( w_k(x) \) are weakly converging \( n \times n \) matrices.
Recall Decomposition Principle for curl.

For sequences \((v_r : \mathbb{R}^n \to \mathbb{R}^n)\) satisfying

\[ v_r \to 0 \text{ weakly in } L^p, \quad \text{curl}v_k \to 0 \text{ in } W^{-1,p} \]
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\]

there exists a decomposition \(v_r = u_r + w_r\), so that:

\[
\text{curl}w_k = 0, \quad \left( \implies \int_{[0,1]^n} L(a + w_k) \geq L(a), \right)
\]
Recall Decomposition Principle for \( \text{curl} \).

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\[
\text{curl} w_k = 0, \quad \left( \implies \int_{[0,1]^n} L(a + w_k) \geq L(a), \right)
\]

and

\[
\|u_k\|_{L^p} \to 0.
\]
Murat’s Constant Rank Hypothesis

The decomposition lemma holds true for CR systems:

\[ A(v) = \sum_{i=1}^{n} A^{(i)}(\partial_i v), \quad v : \mathbb{R}^n \to \mathbb{R}^d, \quad A^{(i)} \in \mathbb{R}^{p \times d}. \]
Murat’s Constant Rank Hypothesis

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whenever the symbol

\[ A(\xi) = \sum_{i=1}^{n} \xi_{i} A^{(i)} \]

satisfies CRH \( \exists C \forall \xi \in S^{n-1} \text{ rk} A(\xi) = C. \)
Murat’s Constant Rank Hypothesis

The decomposition lemma holds true for CR systems:

\[ A(v) = \sum_{i=1}^{n} A^{(i)}(\partial_i v), \quad v : \mathbb{R}^n \to \mathbb{R}^d, \quad A^{(i)} \in \mathbb{R}^{p \times d}. \]

whenever the symbol

\[ A(\xi) = \sum_{i=1}^{n} \xi_i A^{(i)} \text{ satisfies CRH} \quad \exists C \forall \xi \in S^{n-1} \text{ rk}(A(\xi)) = C. \]

The decomposition lemma is false for the following system

\[ A_0(v) = \text{grad}(v) - \text{diag}(\partial_1 v_1, \ldots, \partial_n v_n), \quad v = (v_1, \ldots, v_n), \quad v_i : \mathbb{R}^n \to \mathbb{R}. \]

**Note:** \( A_0(v) = 0 \implies v_i(x) = v_i(x_i). \)
And now....something completely different....

Following Ball-Murat we specialize Morrey’s theorem to **diagonal** matrices

\[ w(x) = \sum_{m=1}^{n} v^{(m)}(x)e_m \otimes e_m, \quad \text{and} \quad L(w) = f(v^{(1)}, \ldots, v^{(n)}). \]
And now....something completely different....

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The **curl for diagonal** matrices $w : \text{curl}(w) \iff \mathcal{A}_0(v)$. 
Following Ball-Murat we specialize Morrey’s theorem to *diagonal* matrices

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**Theorem of Tartar:**
Weak lower semicontinuity of

\[ \int f(v_T(x))dx, \quad A_0(v_T) \text{ pre-compact in } W^{-1,p} \]

implies that \( f \) is separately convex.
And now....something completely different....

Following Ball-Murat we specialize Morrey’s theorem to **diagonal** matrices

\[ \mathbf{w}(x) = \sum_{m=1}^{n} v^{(m)}(x) e_m \otimes e_m, \quad \text{and} \quad L(\mathbf{w}) = f(v^{(1)}, \ldots, v^{(n)}). \]

The **curl for diagonal** matrices \( \mathbf{w} : \text{curl}(\mathbf{w}) \iff A_0(v). \)

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\[ \int f(v_{\tau}(x))dx, \quad A_0(v_{\tau}) \text{ pre – compact in } W^{-1,p} \]

implies that \( f \) is separately convex.

**Problem of Tartar**

Does separate convexity imply weak lower semicontinuity?
Tartar’s Conjecture and its Proof

Theorem (J. Lee, S. Müller, P.F.X.M.)

Let \( f : \mathbb{R}^n \to \mathbb{R}^+ \), separately convex, with, \( 0 \leq f(x) \leq (1 + |x|^p) \).
Theorem (J. Lee, S. Müller, P.F.X.M.)

Let $f : \mathbb{R}^n \to \mathbb{R}^+$, separately convex, with, $0 \leq f(x) \leq (1 + |x|^p)$.

$v_r \rightharpoonup v$ in $L^p$, and $A_0(v_r)$ pre-compact in $W^{-1,p}$.
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Let \( f : \mathbb{R}^n \to \mathbb{R}^+ \), separately convex, with, \( 0 \leq f(x) \leq (1 + |x|^p) \).

\[ v_r \to v \text{ in } L^p, \quad \text{and} \quad A_0(v_r) \text{ pre-compact in } W^{-1,p} \]

implies that for each non-negative testing function \( \varphi \),

\[ \int_{\mathbb{R}^n} f(v(x)) \varphi(x) dx \leq \liminf_{r \to \infty} \int_{\mathbb{R}^n} f(v_r(x)) \varphi(x) dx. \]

Recall. \( A_0(u) = \text{grad}(v) - \text{diag}(\partial_1 v_1, \ldots, \partial_n v_n) \),

\( v = (v_1, \ldots, v_n), \quad v_i : \mathbb{R}^n \to \mathbb{R} \).
The decomposition. (J. Lee, S. Müller, P.F.X.M.)

For sequences \((v_r : \mathbb{R}^n \to \mathbb{R}^n)\) with support in the unit cube satisfying

\[ v_r \rightharpoonup 0 \quad \text{weakly in} \quad L^p, \quad A_0(v_r) \to 0 \quad \text{in} \quad W^{-1,p} \]

there exists a decomposition \(v_r = u_r + w_r\), so that:

1. For each separately convex \(f\) we have Jensen’s Inequality

   \[ \int_{[0,1]} f(a + u_r(x)) \, dx \leq f(a). \]

2. \(k w_r \rightharpoonup 0 \quad \text{in} \quad L^p.\)
The decomposition. (J. Lee, S. Müller, P.F.X.M.)

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For sequences \( (v_r : \mathbb{R}^n \to \mathbb{R}^n) \) with support in the unit cube satisfying

\[
v_r \rightharpoonup 0 \quad \text{weakly in } \mathbb{L}^p, \quad A_0(v_r) \to 0 \quad \text{in } W^{-1,p}
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there exists a decomposition \( v_r = u_r + w_r \), so that:

1. For each separately convex \( f \) we have Jensen’s Inequality

\[
\int_{[0,1]^n} f(a + u_r(x)) dx \geq f(a).
\]

2. \( \|w_r\|_{\mathbb{L}^p} \to 0. \)
The Haar System

Dyadic intervals $\mathcal{D}$ in $\mathbb{R}$ are $[k2^{-n}, (k+1)2^{-n}]$. A Haar function $h_I$ is supported on $I \in \mathcal{D}$ and $h_I = 1$ on the left half of $I$ and $h_I = -1$ on the right half of $I$. 
The Haar System

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Isotropic Haar basis in $L^2(\mathbb{R}^n)$:

$x = (x_1, \ldots, x_n), \quad |I_1| = \cdots = |I_n|, \quad \mathcal{A} = \{0, 1\}^n \setminus \{0\}$.

$$h_{I_1 \times \cdots \times I_n}^{(\varepsilon)}(x) = h_{I_1}^{\varepsilon_1}(x_1) \cdots h_{I_n}^{\varepsilon_n}(x_n), \quad \varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \mathcal{A}. $$
Subsystems of the Haar system

The $j$–th unit vector: $e_j \in \mathbb{R}^n$.

$$\mathcal{H}^j = \{ h^{(e_j)}_{I_1 \times \cdots \times I_n} : |I_1| = \cdots = |I_n| \}.$$
Subsystems of the Haar system

The j–th unit vector: \( e_j \in \mathbb{R}^n \).

\[
\mathcal{H}^j = \{ h_1^{(e_j)}(I_1, \ldots, I_n) \mid |I_1| = \cdots = |I_n| \}.
\]

\( P(e_j) \) is the orthogonal projection onto \( \text{span}\mathcal{H}^{(i)} \).
Subsystems of the Haar system

The j–th unit vector: \( e_j \in \mathbb{R}^n \).

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\( P(e_j) \) is the orthogonal projection onto \( \text{span}\mathcal{H}^{(i)} \).

\[
P(v) = (P^{(e_1)}(v_1), \ldots, P^{(e_n)}(v_n)) \quad v = (v_1, \ldots, v_n)
\]
Subsystems of the Haar system

The j-th unit vector: $e_j \in \mathbb{R}^n$.

$$\mathcal{H}^j = \{ h_{I_1 \times \ldots \times I_n}^{(e_j)} : |I_1| = \cdots = |I_n| \}.$$  

$P(e_j)$ is the orthogonal projection onto $\text{span}\mathcal{H}^j$.

$$P(v) = (P^{(e_1)}(v_1), \ldots, P^{(e_n)}(v_n)) \quad v = (v_1, \ldots, v_n)$$

If $f$ is separately convex then we get Jensen’s inequality on the range of $P$. 
We show:  \[
    f \left( \int_{[0,1]^n} P(v)(x) \, dx \right) \leq \int_{[0,1]^n} f(P(v(x))) \, dx.
\]
We show: \[ f \left( \int_{[0,1]^n} P(v)(x) \, dx \right) \leq \int_{[0,1]^n} f(P(v(x))) \, dx. \]

Since: \[ h_Q^{(e_j)}(x) = h_I(x_j), \quad x \in Q = I_1 \times \cdots \times I_n \]
We show: \[ f \left( \int_{[0,1]^n} P(\nu)(x) \, dx \right) \leq \int_{[0,1]^n} f(P(\nu(x))) \, dx. \]

Since: \[ h_Q^{(e_j)}(x) = h_{l_j}(x_j), \quad x \in Q = l_1 \times \cdots \times l_n \]
Jensen in each variable separately gives
\[
\int_Q f(a_1 + c_1 h_Q^{(e_1)}(x), \ldots, a_n + c_n h_Q^{(e_n)}(x)) \, dx \\
= \int_Q f(a_1 + c_1 h_{l_1}(x_1), \ldots, a_n + c_n h_{l_n}(x_n)) \, dx \geq |Q| f(a).
\]
Separately convex $f$

We show: \[ f \left( \int_{[0,1]^n} P(v)(x) \, dx \right) \leq \int_{[0,1]^n} f(P(v(x))) \, dx. \]

Since: \[ h_Q^{(e_j)}(x) = h_{l_j}(x), \quad x \in Q = l_1 \times \cdots \times l_n \]
Jensen in each variable separately gives

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\int_Q f(a_1 + c_1 h_Q^{(e_1)}(x), \ldots, a_n + c_n h_Q^{(e_n)}(x)) \, dx \\
= \int_Q f(a_1 + c_1 h_1(x_1), \ldots, a_n + c_n h_n(x_n)) \, dx \geq |Q| f(a).
\]

The supports of Haar functions are nested. We may hence iterate.
Directional Haar projection:

For \( \varepsilon \in \mathcal{A} = \{0, 1\}^n \setminus \{0\} \) put

\[
P^{(\varepsilon)} u = \sum_Q \langle u, h_Q^{(\varepsilon)} \rangle h_Q^{(\varepsilon)} |Q|^{-1}.
\]
Directional Haar projection:

For $\varepsilon \in A = \{0, 1\}^n \setminus 0$ put

$$P(\varepsilon) u = \sum_Q \langle u, h_Q^{(\varepsilon)} h_Q^{(\varepsilon)} |Q|^{-1}. $$

**Theorem:** (J. Lee, S. Müller, P.F.X.M.)

If $\varepsilon = (\varepsilon_1, \ldots \varepsilon_n) \in A$ with $\varepsilon_{i_0} = 1$, then
Directional Haar projection:

For \( \varepsilon \in A = \{0, 1\}^n \setminus 0 \) put

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P^{(\varepsilon)} u = \sum_{Q} \langle u, h_{Q}^{(\varepsilon)} \rangle h_{Q}^{(\varepsilon)} |Q|^{-1}.
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**Theorem:** (J. Lee, S. Müller, P.F.X.M.)

If \( \varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in A \) with \( \varepsilon_{i_0} = 1 \), then

\[\|P^{(\varepsilon)}(u)\|_p \leq A_p \|u\|_p^{1/2} \|R_{i_0}(u)\|_p^{1/2}, \quad 2 \leq p < \infty.\]

and
Directional Haar projection:

For $\varepsilon \in \mathcal{A} = \{0, 1\}^n \setminus 0$ put

$$P^{(\varepsilon)} u = \sum_Q \langle u, h^{(\varepsilon)}_Q \rangle h^{(\varepsilon)}_Q |Q|^{-1}.$$

**Theorem:** (J. Lee, S. Müller, P.F.X.M.)

If $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \mathcal{A}$ with $\varepsilon_{i_0} = 1$, then

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and

$$\|P^{(\varepsilon)}(u)\|_p \leq A_p \|u\|_p^{1/p} \|R_{i_0}(u)\|_p^{1-1/p}, \quad 1 < p \leq 2.$$
Directional Haar projection:

For $\varepsilon \in \mathcal{A} = \{0, 1\}^n \setminus \emptyset$ put $P^{(\varepsilon)} u = \sum_{Q} \langle u, h_Q^{(\varepsilon)} \rangle h_Q^{(\varepsilon)} |Q|^{-1}$.

**Theorem: (J. Lee, S. Müller, P.F.X.M.)**

If $\varepsilon = (\varepsilon_1, \ldots \varepsilon_n) \in \mathcal{A}$ with $\varepsilon_{i_0} = 1$, then

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and

$$\|P^{(\varepsilon)}(u)\|_p \leq A_p \|u\|_p^{1/p} \|R_{i_0}(u)\|_p^{1-1/p}, \quad 1 < p \leq 2.$$ 

Exponents are sharp! For instance,

$$\sup_{w \in L^p} \frac{\|P^{(\varepsilon)}w\|_p}{\|R_{i_0} w\|_p^{1/2+\delta} \|w\|_p^{1/2-\delta}} = \infty, \quad 2 \leq p < \infty.$$
Harvest: $L^p$ Estimates for $(v - P(v))$.

If $v_r \rightarrow 0$ weakly in $L^p$ \ $\mathcal{A}_0(v_r) \rightarrow 0$ in $W^{-1,p}$

then

$$\|v_r - P(v_r)\|_p \rightarrow 0.$$
Havest: $L^p$ Estimates for $(v - P(v))$.

If $v_r \rightharpoonup 0$ weakly in $L^p$ \hspace{1cm} A_0(v_r) \to 0$ in $W^{-1,p}$

then

$$\|v_r - P(v_r)\|_p \to 0.$$  

Indeed

$$\|v - P(v)\|_p \leq C_p \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} \|R_j v^{(i)}\|_p^{1/2} \|v^{(i)}\|_p^{1/2}.$$
Harvest: $L^p$ Estimates for $(v - P(v))$.

If $v_r \rightharpoonup 0$ weakly in $L^p$, $A_0(v_r) \to 0$ in $W^{-1, p}$ then

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Indeed

$$\|v - P(v)\|_p \leq C_p \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} \|R_j v(i)\|_p^{1/2} \|v(i)\|_p^{1/2}. $$

Right hand side is bounded by

$$C_p \|A_0 v\|_{W^{-1, p}}^{1/2} \|v\|_p^{1/2} + \|T(v)\|_p, \quad \text{where } T \text{ is compact.}$$
Harvest: $L^p$ Estimates for $(\nu - P(\nu))$.

If $\nu_r \rightharpoonup 0$ weakly in $L^p$, $\mathcal{A}_0(\nu_r) \to 0$ in $W^{-1,p}$
then
$$\|\nu_r - P(\nu_r)\|_p \to 0.$$ Indeed
$$\|\nu - P(\nu)\|_p \leq C_p \sum_{i=1}^n \sum_{j=1,j \neq i}^n \|R_j \nu^{(i)}\|^{1/2}_p \|\nu^{(i)}\|^{1/2}_p.$$

Right hand side is bounded by
$$C_p \|\mathcal{A}_0 \nu\|^{1/2}_{W^{-1,p}} \|\nu\|^{1/2}_p + \|T(\nu)\|_p,$$
where $T$ is compact.

Riesz transforms satisfy
$$\|R_j w\|_p \leq C \|\partial_j w\|_{W^{-1,p}} + \|T w\|_p.$$
The decomposition made explicit

For sequences \((v_r : \mathbb{R}^n \to \mathbb{R}^n)\) with support in the unit cube satisfying

\[
v_r \to 0 \quad \text{weakly in } L^p, \quad A_0(v_r) \to 0 \quad \text{in } W^{-1,p}
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The decomposition made explicit

For sequences \((v_r : \mathbb{R}^n \rightarrow \mathbb{R}^n)\) with support in the unit cube satisfying

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For each separately convex \(f\)

\[ \int_{[0,1]^n} f(a + P(v_r(x))) \, dx \geq f(a). \]
The decomposition made explicit

For sequences \( (v_r : \mathbb{R}^n \to \mathbb{R}^n) \) with support in the unit cube satisfying

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v_r \to 0 \quad \text{weakly in} \quad L^p, \quad A_0(v_r) \to 0 \quad \text{in} \quad W^{-1,p}
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\int_{[0,1]^n} f(a + P(v_r(x))) \, dx \geq f(a).
\]

2. \[
\|v_r - P(v_r)\|_{L^p} \to 0.
\]
Induced Questions

Induced Questions

Directional projections ranging in UMD spaces. R. Lechner in his Ph. D. at J. Kepler University, Linz proved

\[ T_{L^p}(\mathbb{R}^n, X) \leq \lambda_{k R}^0(u) + \lambda_{k A}^1 T_{L^p}(\mathbb{R}^n, X), \]

where \( T \) is the Rademacher Type of \( L^p(X) \) and \( A = A(p, UMD(X)) \).

The Rademacher Type of \( L^p(X) \) enters explicitly in the exponents.
Induced Questions

Directional projections ranging in UMD spaces.
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Directional projections ranging in UMD spaces.

R. Lechner in his Ph. D. at J. Kepler University, Linz proved

$$\| P^{(\varepsilon)}(u) \|_{L^p(\mathbb{R}^n, X)} \leq A \| u \|_{L^p(\mathbb{R}^n, X)} \left( \frac{1}{T} \right)^{1 - \frac{1}{T}} \| R_{i_0}(u) \|_{L^p(\mathbb{R}^n, X)},$$

where $T =$ Rademacher Type of $L^p(X)$ and $A = A(p, \text{UMD}(X))$. The Rademacher Type of $L^p(X)$ enters explicitly in the exponents.
Induced Questions

Directional projections ranging in UMD spaces.

R. Lechner in his Ph. D. at J. Kepler University, Linz proved

\[
\|P^{(e)}(u)\|_{L^p(\mathbb{R}^n, X)} \leq A \|u\|_{L^p(\mathbb{R}^n, X)} \frac{1}{T} \|R_{i0}(u)\|_{L^p(\mathbb{R}^n, X)},
\]

where \( T = \text{Rademacher Type of } L^p(X) \) and \( A = A(p, \text{UMD}(X)) \).

The Rademacher Type of \( L^p(X) \) enters explicitly in the exponents.

Replace the Haar system by Hölder smooth wavelets (A. Kamont, S. Müller, PFXM).
Induced Questions

Directional projections ranging in **UMD spaces**.

**R. Lechner** in his Ph. D. at J. Kepler University, Linz proved

\[
\|P(\varepsilon)(u)\|_{L^p(\mathbb{R}^n, X)} \leq A \|u\|_{L^p(\mathbb{R}^n, X)} \left( \frac{1}{T} \right) \left( \frac{1}{R_{i0}(u)} \right)^{1 - \frac{1}{T}},
\]

where \( T = \text{Rademacher Type of } L^p(X) \) and \( A = A(p, \text{UMD}(X)) \).

The Rademacher Type of \( L^p(X) \) enters explicitly in the exponents.

Replace the Haar system by **Hölder smooth wavelets** (A. Kamont, S. Müller, PFXM).
Admissible wavelet systems (1).

$S$ is the collection of all dyadic cubes in $\mathbb{R}^n$ and
$A = \{ \varepsilon \in \{0,1\}^n : \varepsilon \neq (0, \ldots, 0) \}.$
Admissible wavelet systems (1).

$\mathcal{S}$ is the collection of all dyadic cubes in $\mathbb{R}^n$ and

$\mathcal{A} = \{ \varepsilon \in \{0,1\}^n : \varepsilon \neq (0, \ldots, 0) \}$. Fix an orthonormal basis in $L^2(\mathbb{R}^n)$

$$\{ \varphi_Q^{(\varepsilon)}/\sqrt{|Q|} : Q \in \mathcal{S}, \varepsilon \in \mathcal{A} \}$$
Admissible wavelet systems (1).

$S$ is the collection of all dyadic cubes in $\mathbb{R}^n$ and 
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We assume that $\varphi_Q^{(\varepsilon)}$ satisfies:

- Localization with decay estimates around $Q$
Admissible wavelet systems (1).

\( \mathcal{S} \) is the collection of all dyadic cubes in \( \mathbb{R}^n \) and 
\( \mathcal{A} = \{ \varepsilon \in \{0,1\}^n : \varepsilon \neq (0, \ldots, 0) \} \). Fix an orthonormal basis in \( L^2(\mathbb{R}^n) \)

\[ \left\{ \varphi_{Q}^{(\varepsilon)} / \sqrt{|Q|} : Q \in \mathcal{S}, \varepsilon \in \mathcal{A} \right\} \]

We assume that \( \varphi_{Q}^{(\varepsilon)} \) satisfies:

- Localization with decay estimates around \( Q \)
- Hölder continuity of order \( \alpha \) with \( 0 < \alpha \leq 1 \).
Admissible wavelet systems (1).

$S$ is the collection of all dyadic cubes in $\mathbb{R}^n$ and
$\mathcal{A} = \{ \varepsilon \in \{0,1\}^n : \varepsilon \neq (0, \ldots, 0) \}$. Fix an orthonormal basis in $L^2(\mathbb{R}^n)$

$$\{ \varphi_Q^{(\varepsilon)}/\sqrt{|Q|} : Q \in S, \varepsilon \in \mathcal{A} \}$$

We assume that $\varphi_Q^{(\varepsilon)}$ satisfies:

- Localization with decay estimates around $Q$
- Hölder continuity of order $\alpha$ with $0 < \alpha \leq 1$.
- Sectional oscillation for $i \in \{ j \leq n : \varepsilon_j = 1 \}$.

Thus $\varphi_Q^{(\varepsilon)}$ is a spread-out, Hölder-smooth version of $h_Q^{(\varepsilon)}$. 
Admissible wavelets (2).

We require decay, Hölder estimates and sectional oscillation:

\[
|Q(x)| \lesssim C \downarrow^{1+\text{dist}(x, Q)} s(Q) \alpha_n (1+.)
\]

\[
|Q(t)| \lesssim C \downarrow^{1+\text{dist}(x, Q)} s(Q) \alpha_n (1+.)
\]

Sectional oscillation for

\[
|E_i(f)(x)| \lesssim C \downarrow^{1+\text{dist}(x, Q)} s(Q) \alpha_n (1+.).
\]

where

\[
E_i(f)(x) = R_x \int f(x_1, \ldots, s, \ldots, x_n) \, ds.
\]
Admissible wavelets (2).

We require decay, Hölder estimates and sectional oscillation:

\[ |\varphi_Q^{(\varepsilon)}(x)| \leq C \left( 1 + \frac{\text{dist}(x,Q)}{s(Q)} \right)^{-n(1+\delta)} \]
Admissible wavelets (2).

We require decay, Hölder estimates and sectional oscillation:

- \[ |φ_Q^{(ε)}(x)| \leq C \left( 1 + \frac{\text{dist}(x,Q)}{s(Q)} \right)^{-n(1+δ)} \]

- \[ |φ_Q^{(ε)}(x) - φ_Q^{(ε)}(t)| \leq Cs(Q)^{-α} |x - t|^{α} \left( 1 + \frac{\text{dist}(x,Q)}{s(Q)} \right)^{-n(1+δ)} \]
Induced Questions  Wavelet Projections and Riesz Transforms

Admissible wavelets (2).

We require decay, Hölder estimates and sectional oscillation:

- \[ |\varphi_Q^{(\varepsilon)}(x)| \leq C \left(1 + \frac{\text{dist}(x,Q)}{s(Q)}\right)^{-n(1+\delta)} \]
- \[ |\varphi_Q^{(\varepsilon)}(x) - \varphi_Q^{(\varepsilon)}(t)| \leq Cs(Q)^{-\alpha}|x - t|^{\alpha} \left(1 + \frac{\text{dist}(x,Q)}{s(Q)}\right)^{-n(1+\delta)} \]
- Sectional oscillation for \( i \in \{j \leq n : \varepsilon_j = 1\} \):
  \[ |E_i(\varphi_Q^{(\varepsilon)})(x)| \leq Cs(Q) \left(1 + \frac{\text{dist}(x,Q)}{s(Q)}\right)^{-n(1+\delta)}, \]
Admissible wavelets (2).

We require decay, Hölder estimates and sectional oscillation:

1. \( |\varphi_Q^{(\varepsilon)}(x)| \leq C \left( 1 + \frac{\text{dist}(x,Q)}{s(Q)} \right)^{-n(1+\delta)} \)

2. \( |\varphi_Q^{(\varepsilon)}(x) - \varphi_Q^{(\varepsilon)}(t)| \leq Cs(Q)^{-\alpha} |x - t|^\alpha \left( 1 + \frac{\text{dist}(x,Q)}{s(Q)} \right)^{-n(1+\delta)} \)

3. Sectional oscillation for \( i \in \{ j \leq n : \varepsilon_j = 1 \} : \)
   \[
   |E_i(\varphi_Q^{(\varepsilon)})(x)| \leq Cs(Q) \left( 1 + \frac{\text{dist}(x,Q)}{s(Q)} \right)^{-n(1+\delta)},
   \]
   where
   \[
   E_i(f)(x) = \int_{-\infty}^{x_i} f(x_1, \ldots, s, \ldots, x_n) ds,
   \]
Directional wavelet projections

For $\varepsilon \in \mathcal{A} = \{0, 1\}^n \setminus \emptyset$ put

\[ W^{(\varepsilon)}(u) = \sum_{Q \in \mathcal{S}} \langle u, \varphi^{(\varepsilon)}_Q \rangle \varphi^{(\varepsilon)}_Q |Q|^{-1}, \]
Directional wavelet projections

For \( \varepsilon \in \mathcal{A} = \{0, 1\}^n \setminus \emptyset \) put

\[
W^{(\varepsilon)}(u) = \sum_{Q \in \mathcal{S}} \langle u, \varphi_Q^{(\varepsilon)} \rangle \varphi_Q^{(\varepsilon)} |Q|^{-1},
\]

**Theorem:** (S. Müller, P.F.X.M.)

If \( \varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \mathcal{A} \) with \( \varepsilon_{i_0} = 1 \), then
Directional wavelet projections

For $\varepsilon \in A = \{0, 1\}^n \setminus 0$ put $W^{(\varepsilon)}(u) = \sum_{Q \in \mathcal{S}} \langle u, \varphi^{(\varepsilon)}_Q \rangle \varphi^{(\varepsilon)}_Q |Q|^{-1},$

**Theorem:** (S. Müller, P.F.X.M.)
If $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in A$ with $\varepsilon_i_0 = 1$, then

$$\|W^{(\varepsilon)}(u)\|_p \leq A\|u\|_p^{1-\alpha}\|R^{(\varepsilon)}_{i_0}(u)\|_p\|^{\alpha}, \quad 0 < \alpha < 1.$$
Directional wavelet projections

For $\varepsilon \in A = \{0, 1\}^n \setminus 0$ put

$$W^{(\varepsilon)}(u) = \sum_{Q \in S} \langle u, \varphi^{(\varepsilon)}_Q \rangle \varphi^{(\varepsilon)}_Q |Q|^{-1},$$

**Theorem: (S. Müller, P.F.X.M.)**

If $\varepsilon = (\varepsilon_1, \ldots \varepsilon_n) \in A$ with $\varepsilon_{i_0} = 1$, then

$$\|W^{(\varepsilon)}(u)\|_p \leq A \|u\|_{p}^{1-\alpha} \|R_{i_0}(u)\|_{p}^{\alpha}, \quad 0 < \alpha < 1.$$ 

If $\alpha = 1$, then

$$\|W^{(\varepsilon)}(u)\|_p \leq A \left(1 + \log \frac{\|u\|_p \|R_{i_0}(u)\|_p}{\|R_{i_0}(u)\|_p} \right) \|R_{i_0}(u)\|_p,$$
Directional wavelet projections

For $\varepsilon \in \mathcal{A} = \{0, 1\}^n \setminus 0$ put

$$W^{(\varepsilon)}(u) = \sum_{Q \in S} \langle u, \varphi_Q^{(\varepsilon)} \rangle \varphi_{Q|Q}^{(\varepsilon)} |Q|^{-1},$$

Theorem: (S. Müller, P.F.X.M.)

If $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \mathcal{A}$ with $\varepsilon_{i_0} = 1$, then

$$||W^{(\varepsilon)}(u)||_p \leq A ||u||_p^{1-\alpha} ||R_{i_0}(u)||_p^\alpha, \quad 0 < \alpha < 1.$$ 

If $\alpha = 1$, then

$$||W^{(\varepsilon)}(u)||_p \leq A \left(1 + \log \frac{||u||_p ||R_{i_0}(u)||_p}{||R_{i_0}(u)||_p} \right) ||R_{i_0}(u)||_p,$$

Haar projections are not limiting cases of the above. Check the exponents!
Dyadic decomposition of Haar Projections:

We resolve the discontinuities of the Haar system step by step. At each
dyadic scale we relate the Riesz transforms to Haar projections.
\( b \in C^\infty(\mathbb{R}) \), with support in \([-1, 1]\), so that

\[
\begin{align*}
  b(t) &= b(-t), \quad \text{Lip}(b) \leq 8, \quad \text{and} \quad \int_{-1}^{+1} b(t) dt = 1. \\
  d_0(x) &= 2^n b(2x_1) \cdots b(2x_n) - b(x_1) \cdots b(x_n). \\
  u &= \sum_{\ell=-\infty}^{\infty} u \ast d_\ell, \quad d_\ell(x) = d_0(2^\ell x)2^{n\ell}.
\end{align*}
\]
Decomposing $P^{(\varepsilon)}(u)$

$$S_j = \{ Q \in S : |Q| = 2^{-j} \}, \quad \Delta_{j+\ell}(h_Q^{(\varepsilon)}) = h_Q^{(\varepsilon)} * d_{\ell}.$$
Decomposing $P^{(\varepsilon)}(u)$

$$S_j = \{ Q \in S : |Q| = 2^{-j} \}, \quad \Delta_{j+\ell}(h^{(\varepsilon)}_Q) = h^{(\varepsilon)}_Q \ast d_\ell.$$ 

$$T^{(\varepsilon)}_\ell u = \sum_{j \in \mathbb{Z}} \sum_{Q \in S_j} \langle u, \Delta_{j+\ell}(h^{(\varepsilon)}_Q) \rangle h^{(\varepsilon)}_Q |Q|^{-1}, \quad P^{(\varepsilon)}(u) = \sum_{\ell = -\infty}^{\infty} T^{(\varepsilon)}_\ell u.$$
Decomposing $P^{(\varepsilon)}(u)$

$$S_j = \{ Q \in S : |Q| = 2^{-j} \}, \quad \Delta_{j+\ell}(h_Q^{(\varepsilon)}) = h_Q^{(\varepsilon)} \ast d_\ell.$$  

$$T^{(\varepsilon)}_\ell u = \sum_{j \in \mathbb{Z}} \sum_{Q \in S_j} \langle u, \Delta_{j+\ell}(h_Q^{(\varepsilon)}) \rangle h_Q^{(\varepsilon)} |Q|^{-1}, \quad P^{(\varepsilon)}(u) = \sum_{\ell = -\infty}^{\infty} T^{(\varepsilon)}_\ell u.$$  

The inverse $R_{i_0}^{-1}$ has symbol  

$$\frac{|\xi|}{\xi_{i_0}} = \frac{\xi_{i_0}}{|\xi|} + \sum_{i=1, i \neq i_0}^{\infty} \frac{\xi_i}{\xi_{i_0}} \cdot \frac{\xi_i}{|\xi|}.$$
Decomposing $P^{(\varepsilon)}(u)$

$$S_j = \{ Q \in \mathcal{S} : |Q| = 2^{-j} \}, \quad \Delta_{j+\ell}(h_Q^{(\varepsilon)}) = h_Q^{(\varepsilon)} \ast d_\ell.$$

$$T_\ell^{(\varepsilon)}u = \sum_{j \in \mathbb{Z}} \sum_{Q \in S_j} \langle u, \Delta_{j+\ell}(h_Q^{(\varepsilon)}) \rangle h_Q^{(\varepsilon)} |Q|^{-1}, \quad P^{(\varepsilon)}(u) = \sum_{\ell=-\infty}^{\infty} T_\ell^{(\varepsilon)}u.$$

The inverse $R_{i_0}^{-1}$ has symbol

$$\frac{\xi}{\xi_{i_0}} = \frac{\xi_{i_0}}{|\xi|} + \sum_{i=1, i \neq i_0} \frac{\xi_i}{\xi_{i_0}} \cdot \frac{\xi_i}{|\xi|}.$$

$$T_\ell^{(\varepsilon)}R_{i_0}^{-1}u = T_\ell^{(\varepsilon)}R_{i_0}u + \sum_{i=1, i \neq i_0} T_\ell^{(\varepsilon)}E_{i_0} \partial_i R_i.$$
Fix $p \geq 2$. We have the Norm-Estimates

\[ \| T_\ell^{(\epsilon)} \|_p \leq 2^{-\ell/2}, \quad \| T_\ell^{(\epsilon)} R_{i_0}^{-1} \|_p \leq 2^{+\ell/2}; \quad \ell > 0, \]

\[ \| T_\ell^{(\epsilon)} \|_p \leq 2^{-|\ell|}, \quad \| T_\ell^{(\epsilon)} R_{i_0}^{-1} \|_p \leq 2^{-|\ell|/p}; \quad \ell \leq 0, \]

which imply interpolatory estimates.
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which imply interpolatory estimates

$$\| P^{(\ell)}(u) \|_p \leq C \| R_i u \|^{1/2} \| u \|^{1/2} .$$

as follows
Pattern of Proof

Apply triangle inequality

$$\|P^{(\varepsilon)}(u)\|_p \leq \sum_{\ell=-\infty}^{\infty} \|T^{(\varepsilon)}_{\ell} u\|_p$$
Pattern of Proof

Apply triangle inequality

$$\|P^{(\varepsilon)}(u)\|_p \leq \sum_{\ell=-\infty}^{\infty} \|T^{(\varepsilon)}_{\ell} u\|_p$$

Choose $M$

$$2^M \leq \frac{\|u\|_p \|R_{i_0}\|_p}{\|R_{i_0} u\|_p} \leq 2^{M+1},$$

and split the series at $\ell = M$
Pattern of Proof

Apply triangle inequality

\[ \| P^{(\varepsilon)}(u) \|_p \leq \sum_{\ell = -\infty}^{\infty} \| T_{\ell}^{(\varepsilon)} u \|_p \]

Choose \( M \)

\[ 2^M \leq \frac{\| u \|_p \| R_{i_0} \|_p}{\| R_{i_0} u \|_p} \leq 2^{M+1} , \]

and split the series at \( \ell = M \)

\[ \sum_{\ell = -\infty}^{M} \| T_{\ell}^{(\varepsilon)} R_{i_0}^{-1} \|_p \| R_{i_0} u \|_p + \sum_{\ell = M+1}^{\infty} \| T_{\ell}^{(\varepsilon)} \|_p \| u \|_p \]
Pattern of Proof

Apply triangle inequality

\[ \| P^{(\varepsilon)}(u) \|_p \leq \sum_{\ell=-\infty}^{\infty} \| T^{(\varepsilon)}_\ell u \|_p \]

Choose \( M \)

\[ 2^M \leq \frac{\| u \|_p \| R_{i_0} \|_p}{\| R_{i_0} u \|_p} \leq 2^{M+1}, \]

and split the series at \( \ell = M \)

\[ \sum_{\ell=-\infty}^{M} \| T^{(\varepsilon)}_\ell R^{-1}_{i_0} \|_p \| R_{i_0} u \|_p + \sum_{\ell=M+1}^{\infty} \| T^{(\varepsilon)}_\ell \|_p \| u \|_p \]

Insert the norm estimates for the operators
Pattern of Proof

Apply triangle inequality

\[ \| P^{(\varepsilon)}(u) \|_p \leq \sum_{\ell = -\infty}^{\infty} \| T^{(\varepsilon)}_{\ell} u \|_p \]

Choose \( M \)

\[ 2^M \leq \frac{\| u \|_p \| R_{i_0} \|_p}{\| R_{i_0} u \|_p} \leq 2^{M+1}, \]

and split the series at \( \ell = M \)

\[ \sum_{\ell = -\infty}^{M} \| T^{(\varepsilon)}_{\ell} R_{i_0}^{-1} \|_p \| R_{i_0} u \|_p + \sum_{\ell = M+1}^{\infty} \| T^{(\varepsilon)}_{\ell} \|_p \| u \|_p \]

Insert the norm estimates for the operators

\[ 2^{M/2} \| R_{i_0} u \|_p + 2^{-M/2} \| u \|_p \leq C \| R_i u \|^{1/2}_p \| u \|^{1/2}_p. \]
Decomposing $W^{(\epsilon)}(u)$

A similar decomposition based on Calderon’s reproducing formula gives a decomposition of the wavelet projection

$$W^{(\epsilon)}(u) = \sum_{Q \in S} \langle u, \varphi_Q^{(\epsilon)} \rangle \varphi_Q^{(\epsilon)} |Q|^{-1}.$$
Decomposing $W^{(\varepsilon)}(u)$

A similar decomposition based on Calderon's reproducing formula gives a decomposition of the wavelet projection

$$W^{(\varepsilon)}(u) = \sum_{Q \in S} \langle u, \varphi^{(\varepsilon)}_Q \rangle \varphi^{(\varepsilon)}_Q |Q|^{-1}.$$  

With $T^{(\varepsilon)}_\ell u = \sum_{j \in \mathbb{Z}} \sum_{Q \in S_j} \langle u, \Delta_{j+\ell}(\varphi^{(\varepsilon)}_Q) \rangle \varphi^{(\varepsilon)}_Q |Q|^{-1}$, $\Delta_{j+\ell}(\varphi^{(\varepsilon)}_Q) = \varphi^{(\varepsilon)}_Q * d_\ell$.

we get the decomposition

$$W^{(\varepsilon)}(u) = \sum_{\ell = -\infty}^{\infty} T^{(\varepsilon)}_\ell u.$$
Norm-Estimates for the decomposing operators

\[ \| T^{(\epsilon)}_{\ell} \|_p \leq C 2^{-\ell \alpha}, \quad \| T^{(\epsilon)}_{\ell} R^{-1}_{i_0} \|_p \leq 2^{\ell - \ell \alpha}; \quad \ell > 0, \]
\[ \| T^{(\epsilon)}_{\ell} \|_p \leq 2^{-|\ell| |\ell|}, \quad \| T^{(\epsilon)}_{\ell} R^{-1}_{i_0} \|_p \leq 2^{-|\ell| |\ell|}; \quad \ell \leq 0, \]

imply
Norm-Estimates for the decomposing operators

\[ \| T^{(e)}_\ell \|_p \leq C2^{-\ell \alpha}, \quad \| T^{(e)}_\ell R_{i_0}^{-1} \|_p \leq 2^{\ell-\ell \alpha}; \quad \ell > 0, \]
\[ \| T^{(e)}_\ell \|_p \leq 2^{-|\ell|} |\ell|, \quad \| T^{(e)}_\ell R_{i_0}^{-1} \|_p \leq 2^{-|\ell|} |\ell|; \quad \ell \leq 0, \]

imply

\[ \| W^{(e)}(u) \|_p \leq A \| u \|_p^{1-\alpha} \| R_{i_0}(u) \|_p^\alpha, \quad 0 < \alpha < 1. \]
Norm-Estimates for the decomposing operators

\[ \| T^{(\varepsilon)}_{\ell} \|_p \leq C 2^{-\ell \alpha}, \quad \| T^{(\varepsilon)}_{\ell} R_{i_0}^{-1} \|_p \leq 2^{\ell - \ell \alpha}; \quad \ell > 0, \]
\[ \| T^{(\varepsilon)}_{\ell} \|_p \leq 2^{-|\ell|} \| \ell \|, \quad \| T^{(\varepsilon)}_{\ell} R_{i_0}^{-1} \|_p \leq 2^{-|\ell|} \| \ell \|; \quad \ell \leq 0, \]

imply
\[ \| W^{(\varepsilon)}(u) \|_p \leq A \| u \|_p^{1-\alpha} \| R_{i_0}(u) \|_p^\alpha, \quad 0 < \alpha < 1, \]
and
\[ \| W^{(\varepsilon)}(u) \|_p \leq A \left( 2 + \log \frac{\| u \|_p \| R_{i_0} \|_p}{\| R_{i_0}(u) \|_p} \right) \| R_{i_0}(u) \|_p, \quad \alpha = 1. \]
Norm estimates for $T^{(ε)}_{ℓ}$, $T^{(ε)}_{ℓ} R^{-1}_{i_0}$. Reduction to permutation operators and to projections onto block bases of the Haar system:

$$T^{(ε)}_{ℓ} u = \sum_{j \in \mathbb{Z}} \sum_{Q \in S_j} \langle u, \Delta_{j+ℓ}(h^{(ε)}_Q) \rangle h^{(ε)}_Q |Q|^{-1}.$$  

$ℓ > 0$ $→$ projections. $ℓ < 0$ $→$ rearrangements. 

$Q \in S_j, \quad |Q| = 2^{-nj}$. $D^{(ε)}(Q)$ discontinuities of $h^{(ε)}_Q$.

$$D^{(ε)}_{j+ℓ}(Q) = \{ z : d(z, D^{(ε)}(Q)) \leq 2^{-(j+ℓ)} \}$$  

strips of width $2^{-(j+ℓ)}$ around the discontinuities.

$\Delta_{j+ℓ}(h^{(ε)}_Q)$ lives on $D^{(ε)}_{j+ℓ}(Q)$ and oscillates at scale $\sim 2^{-ℓ} diam Q$.
\[ |\Delta_{j+\ell}(h_Q^{(\varepsilon)})| \leq C, \quad \text{Lip}(\Delta_{j+\ell}(h_Q^{(\varepsilon)})) \leq C2^\ell / \text{diam}Q. \]

Cover \( Q \cap D_{j+\ell}^{(\varepsilon)}(Q) \) with pairwise disjoint cubes of diameter \( 2^{-\ell} \text{diam}Q \).
\[ \rightarrow \{ E_1(Q), \ldots, E_M(Q) \}, \quad M \leq C2^{n(\ell-1)}. \]

Form \( d_Q = \sum_{i=1}^{M} h_{E_i(Q)} \) and \( G(u) = \sum_{j \in \mathbb{Z}} \sum_{Q \in S_j} \langle u, h_Q^{(\varepsilon)} \rangle d_Q |Q|^{-1} \),
and compare with
\[ \mathcal{T}_\ell^{(\varepsilon)*} u = \sum_{j \in \mathbb{Z}} \sum_{Q \in S_j} \langle u, h_Q^{(\varepsilon)} \rangle \Delta_{j+\ell}(h_Q^{(\varepsilon)}) |Q|^{-1}. \]
We have
\[ \| T_\ell^{(\varepsilon)} u \|_q \leq C \| G(u) \|_q, \]
and
\[ \| (T_\ell^{(\varepsilon)} R_{i_0}^{-1})^* u \|_q \leq C 2^\ell \| G(u) \|_q, \]
Estimates for the projection itself,
\[
\| G(u) \|_q \leq 2^{-\ell/2} \| u \|_q \quad \text{for} \quad q \leq 2.
\]
\[
\| G(u) \|_q \leq 2^{-\ell/p} \| u \|_q \quad \text{for} \quad q \geq 2.
\]