Journé commutators and characterisations of product BMO

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What we are doing

We study exact symbol classes so that commutators

\[ [J_1, ...[J_t, b]...] \]

are bounded in, say, \( L^2 \).

Here \( b \) stands for multiplication by \( b \) and the \( J \) are Journé operators in products of Euclidean spaces. The latter are multi parameter siblings of Calderon Zygmund operators, as seen before.

Simplest examples of such \( J \) are tensor products of Riesz transforms or multiple Hilbert transforms.

We will see that these simple representants are the correct testing class for boundedness of the commutator.
A Hankel operator with symbol $b$ is

$$H_b : L^2_+ \rightarrow H^2_-, \ f \mapsto P_- b P_+ f$$

A Toeplitz operator with symbol $b$ is

$$T_b : L^2_+ \rightarrow H^2_+, \ f \mapsto P_+ b P_+ f$$
Nehari Theorem

**Theorem**

\( H_b \) is bounded iff there is a bounded function \( \beta \) with \( P_-\beta = P_-b \).

Moreover \( \|H_b\| = \inf_{\beta: P_-\beta = P_-b} \|\beta\|_\infty. \)

In modern terms we have \( \|H_b\| \sim \|P_-b\|_{BMO}. \)
proof of Nehari

\[ \| H_b \| = \sup_{\| g \|_{H^2_\perp} = 1} \sup_{\| f \|_{H^2} = 1} |(H_b f, g)| \]
\[ = \sup_{\| g \|_{H^2} = 1} \sup_{\| f \|_{H^2} = 1} |(P_-(bf), g)| \]
\[ = \sup_{\| g \|_{H^2_\perp} = 1} \sup_{\| f \|_{H^2_+} = 1} |(P_-bf, g)| \]
\[ = \sup_{\| g \|_{H^2} = 1} \sup_{\| f \|_{H^2} = 1} |(P_-b, \bar{f} g)| \]

Anti-analytic part of \( b \) defines bounded linear functional on \( H^1 \subset L^1 \).
Extend by Hahn Banach to all of \( L^1 \), i.e. a bounded function with the same anti-analytic part as \( b \).

Using \( H^1 - BMO \) duality, we get the classification of BMO.
Toeplitz operators

\[ P_+ b P_+ . \]

Clearly

\[ \| T_b \| \leq \| b \|_\infty \]

But \( L^\infty \) also characterises boundedness:

It is easy to see that

\[ \bar{\lambda}^n P_+ \lambda^n \to I \]

in \( L^2 \) in SOT. Nothing happens to such \( f \) with FS cut off at \(-n\). So

\[ \bar{\lambda}^n P_+ b P_+ \lambda^n \to b \]

in \( L^2 \) in SOT.

For trigonometric polynomial \( p \) and \( n \) large enough

\[ \| bp \| \leq \| T_b \| \| p \| \]
Commutators $[H, b]$

$[b, H]f = b \cdot Hf - H(bf)$ where $H$ is the Hilbert transform and $b$ is multiplication by the function $b$.

If we write

$H = P_+ - P_-$ and $I = P_+ + P_-$

then

$[b, H] = [(P_+ + P_-)b, (P_+ - P_-)] = 2P_- bP_+ - 2P_+ bP_-$,

two Hankel operators with orthogonal domains and ranges.

**Theorem (Nehari)**

$[b, H]$ bounded in $L^2$ iff $b \in BMO$. 
Commutators $[H_1 H_2, b]$ with $b(x_1, x_2)$

Tensor product one-parameter case.

**Theorem (Ferguson, Sadosky)**

$[H_1 H_2, b]$ bounded in $L^2$ iff $b \in \text{bmo}$ ‘little BMO’

$$||b||_{bmo} = \max\{\sup_{x_1} ||b(x_1, \cdot)||_{\text{BMO}}, \sup_{x_2} ||b(\cdot, x_2)||_{\text{BMO}}\}$$

i.e. $b$ uniformly in BMO in each variable separately or of bounded mean oscillation on rectangles.

more Hilbert transforms $[H_1 H_2 H_3, b]$ etc implicit.
Commutators $[H_1, [H_2, b]]$ with $b(x_1, x_2)$

**Theorem (Ferguson, Lacey)**

$[H_1, [H_2, b]]$ bounded in $L^2$ iff $b \in BMO$ ‘product BMO’

\[
\|b\|_{BMO}^2 = \sup_O \frac{1}{|O|} \sum_{R \subset O} |(b, h_R)|^2
\]

more iterations $[H_1, [H_2, [H_3, b]]]$ ‘not’ implicit, but Terwilliger, Lacey.
Commutators $[H_2, [H_3 H_1, b]]$ with $b(x_1, x_2, x_3)$

**Theorem (Ou, Strouse, P.)**

$[H_2, [H_3 H_1, b]]$ bounded in $L^2$ iff $b \in BMO_{(13)2}$ ‘little product BMO’

$$||b||_{BMO_{(13)2}} = \max \{ \sup_{x_1} ||b(x_1, \cdot, \cdot)||_{BMO}, \sup_{x_3} ||b(\cdot, \cdot, x_3)||_{BMO} \}$$

$b$ uniformly in product BMO when fixing variables $x_3$ and $x_1$.

more Hilbert transforms and more iterations implicit.
Commutators $[H_2, [H_3 H_1, b]]$ with $b(x_1, x_2, x_3)$

Infact TFAE:

1. $b \in BMO_{(13)_2}$
2. $[H_2, [H_1, b]]$ and $[H_2, [H_3, b]]$ bounded in $L^2(T^3)$
3. $[H_2, [H_3 H_1, b]]$ bounded in $L^2(T^3)$.

1 eq 2: Wiener’s theorem and Ferguson, Lacey:

$$[H_2, [H_1, b]] f(x_1, x_2) g(x_3) = g(x_3) [H_2, [H_1, b]] f(x_1, x_2)$$

2 eq 3: Toeplitz argument: typical terms that arise:

$$P_1^+ P_2^+ b P_1^- P_2^- \quad \text{and} \quad P_1^+ P_2^+ P_3^+ b P_1^- P_2^- P_3^+$$
Riesz commutators
and the absence of splitting into Hankels.
One parameter: \([R_i, b]\)

It is a classical result by Coifman, Rochberg and Weiss, that the Riesz transform commutators classify \(BMO\). For each symbol \(b \in BMO\) we may choose the worst Riesz transform. In this sense

\[
\| [b, R_i] \|_{2 \to 2} \lesssim \| b \|_{BMO}
\]

But

\[
\| b \|_{BMO} \lesssim \sup_i \| [b, R_i] \|_{2 \to 2}
\]

Testing class for CZOs.
One parameter tensor product: $[R_{1,i_1} R_{2,i_2}, b]$

Through use of the little BMO norm, one sees that

$$\| [b, R_{1,i_1} R_{2,i_2}] \|_{2 \to 2} \lesssim \| b \|_{bmo}$$

Modifying the Coifman, Rochberg and Weiss proof one also sees

$$\| b \|_{bmo} \lesssim \sup_{i_1, i_2} \| [b, R_{1,i_1} R_{2,i_2}] \|_{2 \to 2}$$
multi-parameter: \([R_{2,j_2}, [R_{1,j_1}, b]]\)

**Theorem (Lacey, Pipher, Wick, P.)**

\[
\sup_{j_1,j_2} \|[R_{2,j_2}, [R_{1,j_1}, b]]\| \sim \|b\|_{BMO}.
\]

*By BMO, we mean Chang–Fefferman product BMO.*

Implicit generalizations with similar proof.

Testing for CZOs.
multi-parameter tensor product: \([R_{2,j_2}, [R_{1,j_1} R_{3,j_3}, b]]\)

**Theorem (Ou, Strouse, P.)**

\[ \| b \|_{BMO_{(13)2}} \sim \|[R_{2,j_2}, [R_{1,j_1} R_{3,j_3}, b]]\| \]

*where we mean little product BMO.*

Implicit generalizations but proof is (substantially) more difficult when dimensions are greater than 2 and in the presence of three or more tensor products.

Testing for (paraproduct-free) Journé.

**Theorem (Ou, Strouse, P.)**

\[ \|[J_1, [J_2, b]]\| \leq \| b \|_{BMO} \]

*where we mean little product BMO and paraproduct free Journe operators.*
Lower bounds

Lower estimates: A hard one-parameter argument.

\[ [b, H]b = P_- |P_- b|^2 - P_+ |P_+ b|^2 \]

Due to symmetry of the Fourier supports
\[ \|b\|_4^2 \lesssim \|P_- |P_- b|^2 - P_+ |P_+ b|^2\|_2 \leq \|[b, H]\|_{2 \rightarrow 2} \|b\|_2 \]

If \( b \) is supported on \( I \)
\[ \|b\|_2 \leq |I|^{1/4} \|b\|_4 \lesssim |I|^{1/4} \|[b, H]\|_{2 \rightarrow 2} \|b\|_2^{1/2} \]

This is a lower BMO estimate.
Cones

Product of 1-dimensional spaces:
Hilbert transform is analytic projection. This is heavily used when testing on $\bar{b}$.

Product of $n > 1$-dimensional spaces:
no obvious replacement since half plane projections are not CZO.

Their discontinuity on the Fourier side is too large.
Replace $H$ by well chosen, smooth half plane projections that are CZO.
Cones

How to lean on Hankel operators.

\[ [C, b]f = C(bf) \] if \( f \) is so that \( C \)'s and \( f \)'s Fourier symbol have separate support.

Compare: \( P_–bP_+ \).

How does this help to pass to Riesz transforms?
Polynomials

Observe: \( [T_1 T_2, b] = T_1[T_2, b] + [T_1, b] T_2 \)

If the commutator with \( T_1 T_2 \) is large, then one of the commutators with \( T_i \) has to be large.

This generalizes to polynomials if one controls degrees, coefficients, operator norms....

Riesz transforms have Fourier symbols \( \xi_i \) on \( S^n \) (monomials) well adapted for polynomial approximation.
Cones and iterated tensor products

To provide a lower estimate

$$\sup_{i_1,i_2,i_3} \| [C_{2,i_2}, [C_{3,i_3} C_{1,i_1}, b]] \| \geq \| b \|_{BMO_{(13)2}}$$

we have to choose the ideal three cones for $b$ and then an appropriate function to test on.

Two cones stem from the iterated case and are very specific, of a specific direction and large aperture. The third one is more flexible and allows to blackbox previous lower estimates via a Toeplitz type argument:
Passage to tensor products of Riesz transforms

The difficulties that arise from now on are seen when there are no iterates. We treat for simplicity an argument with no iterates present that generalises.

Tensor products of cone operators in $\mathbb{R}^2$ and their testing functions are illustrated by the following picture.
Passage to tensor products of Riesz transforms

What goes wrong:

If \([R^2_{1,i_1} R^2_{2,i_2}, b]\) large, cannot say \([R_{1,i_1} R_{2,i_2}, b]\) remains large.

It is too wasteful to just have any lower estimates of tensor products of cone operators.

Need a lower estimate using operators that can be written as polynomial in tensor products of Riesz transforms, so all have the same degree:

If \([R^2_{1,i_1} R^2_{2,i_2}, b]\) large, can say \([R_{1,i_1} R_{2,i_2}, b]\) remains large.
Upper estimates

Need a stability estimate for Journe commutators to make the argument work:

$$\|[J_1, \ldots, [J_t, b], \ldots, ]\| \leq C\|b\|$$

with $C \to 0$ when defining constants of $J$ do.

This is a straightforward (but long) calculation thanks to Finnish mathematics.

Use very general Haar shift operators for Journe operators that locally look like tensor products. (Martikainen)