Introduction to Continuous Logic
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Exercise 1

1. Suppose M, M', M'' are metric spaces and $f, f_n : M \to M'$ and $f', f'_n : M' \to M''$ are functions ($n < \omega$). Suppose $(f_n)_{n<\omega}$ converges uniformly to f on M and $(f'_n)_{n<\omega}$ converges uniformly to f' on M'. Show that $(f'_n \circ f_n)_{n<\omega}$ converges uniformly to $f' \circ f$ on M.

2. Prove the Stone-Weierstrass theorem: Let X be a compact Hausdorff space containing at least two points, let $I \subset \mathbb{R}$ be an interval and consider $\mathcal{A} = C(X, I)$ the lattice of continuous functions from X to I, with the uniform convergence topology. If $\mathcal{B} \subset \mathcal{A}$ is a sublatice such that for every distinct $x, y \in X$ and every $a, b \in I$ and $\varepsilon > 0$, there is $f \in \mathcal{B}$ such that $|f(x) - a| < \varepsilon$ and $|f(y) - b| < \varepsilon$, then $\mathcal{B}$ is dense in $\mathcal{A}$.

Hint: Given $f \in \mathcal{A}$ and $\varepsilon > 0$, for each pair of points $x, y \in X$, the assumption gives $g_{x,y} \in \mathcal{B}$ approximating f at x and y. Let $V_{x,y} = \{z \in X : f(z) - \varepsilon < g_{x,y}(z)\}$. This is an open neighbourhood of y and we can cover X by $\{V_{x,y} : y \in X\}$ and use compactness to define a g_x such that $f(z) - \varepsilon < g_x(z)$ for all $z \in X$. Then similarly let x vary to take care of the other direction.

3. In classical logic if '$=$' isn't real identity, the identity axioms (see e.g. the material for the course Logic I) are exactly what is needed to be able to define a model on the equivalence classes of $=$. Analogously, if in a metric structure d is not a metric but only a pseudometric (nonnegativity, symmetry and triangle inequality holds for d, but there may be $x \neq y$ with $d(x, y) = 0$) one can define a metric space by considering the equivalence classes of the equivalence relation $d(x, y) = 0$. Show that the uniform continuity demands on functions and predicates guarantee that the symbols from the signature have well-defined interpretations in the quotient structure. (See the first appendix and the section on prestructures in the material for details.)

4. Prove that for each L-term $t(x_1, \ldots, x_n)$ and each L-formula $\varphi(x_1, \ldots, x_n)$ there are functions Δ_t and Δ_φ form $(0, 1]$ to $(0, 1]$ such that for any L-prestructure $\mathcal{M}$, Δ_t is a modulus of uniform continuity for the function $t^{\mathcal{M}} : M^n \to M$ and Δ_φ is a modulus of uniform continuity for the predicate $\varphi^{\mathcal{M}} : M^n \to [0, 1]$.

5. Show that a quotient L-prestructure $\mathcal{M}$ can be extended to an L-structure $\mathcal{N}$, i.e., that the predicates and functions of $\mathcal{M}$ can be extended to the completion of the metric space (M, d) such that the extensions have the same moduli of uniform continuity as the original functions.