Logic Toolbox
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Exercise 7

1. If $f, g \in {}^\omega\omega$, say $f <^* g$ iff there is some $n < \omega$ such that for all $m > n, m < \omega$, $f(m) < g(m)$. Let $\mathcal{F} \subset {}^\omega\omega$ with $|\mathcal{F}| = \kappa$. Assuming $\mathrm{MA}(\kappa)$, show that there is $g \in {}^\omega\omega$ such that for all $f \in \mathcal{F}$, $f <^* g$.

Hint: Let P be the set of pairs (p, F) such that p is a partial function from ω to ω and F is a finite subset of $\mathcal{F}$, $(p, F) \leq (q, G)$ iff $q \subset p$, $G \subset F$ and

$$\forall f \in G \forall n \in (\mathrm{dom}(p) \setminus \mathrm{dom}(q))\, p(n) > f(n).$$

2. Assume $\mathrm{MA}(\kappa)$. Let $\mathcal{A}$ be a family of Lebesgue measurable subsets of $\mathbb{R}$ with $|\mathcal{A}| = \kappa$. Show that $\bigcup \mathcal{A}$ is Lebesgue measurable and $\mu(\bigcup \mathcal{A}) = \mu(\bigcup \mathcal{B})$ for some countable $\mathcal{B} \subset \mathcal{A}$.

3. Assume $\mathrm{MA}(\kappa)$. Let X be a compact c.c.c. Hausdorff space and U_α dense open subsets of X for $\alpha < \kappa$. Then $\bigcap_{\alpha<\kappa} U_\alpha \neq \emptyset$.

Hints: Use the partial order $P = \{p \subset X : p \text{ is open and } p \neq \emptyset\}$ with $p \leq q$ iff $p \subseteq q$. Compactness is equivalent to the claim that any collection of closed sets with the finite intersection property has nonempty intersection. Use dense sets $D_\alpha = \{p \in P : \overline{p} \subset U_\alpha\}$ and the fact that X is regular (see, e.g., Wikipedia).

4. Assume $\mathrm{MA}(\omega_1)$. Let X be a c.c.c. topological space and $\{U_\alpha : \alpha < \omega_1\}$ a family of non-empty open subsets of X. Then there is an uncountable $A \subset \omega_1$, such that $\{U_\alpha : \alpha \in A\}$ has the finite intersection property.

Hints: Let $V_\alpha = \bigcup_{\gamma>\alpha} U_\gamma$. First find α such that for all $\beta > \alpha$, $\overline{V_\beta} = \overline{V_\alpha}$. Then for such an α use the partial order $P = \{p \subset V_\alpha : p \text{ is open and } p \neq \emptyset\}$ and for a suitable G let $A = \{\gamma < \omega_1 : \exists p \in G\, p \subseteq U_\gamma\}$.

5. A partial order P has ω_1 as a *precaliber* iff whenever $p_\alpha \in P$ for $\alpha < \omega_1$, there is an uncountable $X \subseteq \omega_1$ such that $\{p_\alpha : \alpha \in X\}$ has the finite intersection property (for all finite $s \subset X$ $\exists q \forall \alpha \in s\, q \leq p_\alpha$). Show that $\mathrm{MA}(\omega_1)$ implies that every c.c.c. P has ω_1 as a precaliber.

6. Assume $\mathrm{MA}(\omega_1)$. Let A_α be a Lebesgue measurable subset of $\mathbb{R}$ for $\alpha < \omega_1$, with $\mu(A_\alpha) > 0$. Show that for some uncountable $X \subset \omega_1$, $\mu\left(\bigcap_{\alpha\in X} A_\alpha\right) > 0$.

Hint: If $\forall \alpha\, \mu(A_\alpha) > \varepsilon$, let $P = \{s \subset \omega_1 : |s| < \omega \text{ and } \mu\left(\bigcap_{\alpha\in s} A_\alpha\right) > \varepsilon\}$. Show that P has c.c.c. and apply the previous exercise to $\{\{\alpha\} : \alpha < \omega_1\}$.