Logic Toolbox
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Exercise 4

1. Prove Zorn's lemma using transfinite induction.

2. Prove that $\mathbb{R}^3 \setminus \mathbb{Q}^3$ is a union of disjoint lines.

3. For a subset A of the plane $\mathbb{R}^2$, the horizontal section of A generated by $y \in \mathbb{R}$ (i.e., its projection onto the first coordinate) is $A^y = \{x \in \mathbb{R} : (x, y) \in A\}$. The vertical section of A generatec by $x \in \mathbb{R}$ is $A_x = \{y \in \mathbb{R} : (x, y) \in A\}$.

Prove that there exists a subset A of the plane with every horizontal section A^y being dense in $\mathbb{R}$ and with every vertical section A_x having precisely one element. Hint: Enumerate the set $\{(a, b) \times \{y\} : a, b, y \in \mathbb{R}, a < b\}$.

4. Show that $\mathbb{R}^3$ is the union of disjoint circles. Hint: generalize the idea from the line example from the lectures.