Logic Toolbox
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Exercise 3

1. Show that the axiom of choice is equivalent to the statement: If I is a set and for each $i \in I$, X_i is a nonempty set, then the cartesian product $\prod_{i \in I} X_i$ is nonempty.

2. Prove *Cantor's theorem*: For any set x, $x \prec \mathcal{P}(x)$. Hint: generalize Cantor's diagonal argument used to show that there are uncountably many reals.

3. Prove that $\kappa^{\lambda+\mu} = \kappa^{\lambda} \cdot \kappa^{\mu}$.

4. Prove that $(\kappa \cdot \lambda)^{\mu} = \kappa^{\mu} \cdot \lambda^{\mu}$.

5. Prove that $(\kappa^{\lambda})^{\mu} = \kappa^{\lambda \cdot \mu}$.

6. Prove for cardinals κ, λ, μ:

(1) $\kappa \leq \lambda \Rightarrow \kappa + \mu \leq \lambda + \mu$.
(2) $\kappa \leq \lambda \Rightarrow \kappa \cdot \mu \leq \lambda \cdot \mu$.
(3) $\kappa \leq \lambda \Rightarrow \kappa^{\mu} \leq \lambda^{\mu}$.
(4) $0 < \kappa \leq \lambda \Rightarrow \mu^{\kappa} \leq \mu^{\lambda}$.

Why cannot we substitute $<$ for $\leq$ above?