Logic Toolbox
Department of Mathematics and Statistics, University of Helsinki
Fall 2015
Exercise 2

Definition 1. If (X, R) is a well-order and $x \in X$ then the *initial segment of* (X, R) *determined by* x is the well-ordered set

$$\operatorname{seg}_{(X,R)} x = \{y \in X : yRx\},$$

(with its inherited order).

When $(X, R) = (\alpha, \in)$ for some ordinal α we write $\operatorname{seg}_\alpha \beta$ for $\operatorname{seg}_{(\alpha,\in)} \beta$.

1. Let α be an ordinal. Show that $\beta \in \alpha$ iff β is an initial segment of α.

2. Let α be an ordinal and $f : (X, R) \to (\alpha, \in)$ be an isomorphism. Show that the image under f of any initial segment of (X, R) is an ordinal.

3. Prove the following version of the Transfinite Induction Principle:

> Let $P(R)$ be a property of well-orderings. Assume that for every well-ordering S if $P(T)$ holds for every initial segment T of S, then $P(S)$ holds. Then $P(R)$ holds for all well-orderings.

4. Prove that every well-ordered set is isomorphic to a unique ordinal (e.g., using the induction principle from the previous exercise).

The next two exercises prove a special case of *Hartog's Theorem.* Consider the set

$$H := \{[(X, R)]_{\cong} : X \subseteq \mathbb{N} \text{ and } (X, R) \text{ is a well-order}\},$$

where $[(X, R)]_{\cong}$ is the equivalence class of (X, R) under isomorphism. Order H by $[(X, R)]_{\cong} < [(Y, S)]_{\cong}$ iff (X, R) is isomorphic to an initial segment of (Y, S).

5. Show that

(1) $<$ is well-defined on H.
(2) $(H, <)$ is a linear order.
(3) $(H, <)$ is a well-order.

6. Show that

(1) H is closed under initial segments.
(2) H cannot be countable.

Conclude that H must be isomorphic to the first uncountable ordinal.