1.2 Transfinite induction and recursion

Let's recall the induction principle of the natural numbers:

Induction principle 1. Assume $A \subseteq \mathbb{N}$ is as set satisfying

- $0 \in A$,
- if $n \in A$ then $n+1 \in A$.

Then $A = \mathbb{N}$.

This justifies the ordinary *proof by induction*: Take a set $A \subseteq \mathbb{N}$ of natural numbers satisfying some property. Show that $0 \in A$. Then show that whenever n is in A then $n+1$ must also be in A. Conclude that A must be all of $\mathbb{N}$.

Another (equivalent) form of the induction priciple is:

Induction principle 2. Assume $A \subseteq \mathbb{N}$ is as set satisfying

- $0 \in A$,
- for any $n \in N$ if $k \in A$ for all $k < n$ then $n \in A$.

Then $A = \mathbb{N}$.

The property behind these principles is the well-order of $\mathbb{N}$. If one assums towards a contradiction that one of the principles does not hold, i.e., there is some $A \neq \mathbb{N}$ with the given properties, then the least element of $\mathbb{N} \backslash A$ provides a contradiction.

Now the second form of the principle is the one that most easily can be generalized:

Theorem 1.11 (Transfinite induction)**.** *Let $P(\alpha)$ be a property of ordinals. Assume that for all ordinals β if $P(\gamma)$ holds for all $\gamma < \beta$, then $P(\beta)$ holds. Then we have $P(\alpha)$ for all ordianls α.*

Proof. Suppose $P(\alpha)$ fails for some α. Then look at the set

$$X = \{\gamma \leq \alpha : P(\gamma) \text{ fails}\}.$$

X is nonempty, as $\alpha \in X$, so as any set of ordinals is well-ordered, X has a least element β. But $P(\gamma)$ holds for all $\gamma < \beta$ so by assumption $P(\beta)$ holds, a contradiction. □

Sometimes it is convenient to use another formulation of the induction principle:

Theorem 1.12 (Transfinite induciton, second formulation)**.** *Let $P(R)$ be a property of well orderings. Assume that for every well ordring S, if $P(T)$ holds for every initial segment T of S, then $P(S)$. Then $P(R)$ holds for all R.*

Proof. Exercise. □

Using transfinite induction, one can prove further properties of ordinals:

Lemma 1.13. *If α, β are ordinals which are isomorphic as linear orders, then $\alpha = \beta$. Furthermore, the only isomorphism form α to itself is the identity function.*

Proof. Let $P(\alpha)$ be the property

> The only ordinal isomorphic to α is α and the only isomorphism from α to $\alpha$is the identity funciton.

Assume $P(\gamma)$ holds for all $\gamma < \beta$. We need to show that $P(\beta)$ holds. So let $f : \beta \to \delta$ be an isomorphism. Now for each $\gamma < \beta$ $f \restriction \gamma$ is an isomorphism $\gamma \to f(\gamma)$. As $P(\gamma)$ holds, $f(\gamma) = \gamma$, and since this was true for all $\gamma < \beta$, f is the indentity function on β. So $\delta = \beta$ and $P(\beta)$ holds. By transfinite induction, $P(\alpha)$ must hold for all ordinals α. □

One can also see ordinals as canonical representatives for well-orders:

Lemma 1.14. *Every well-ordered set is isomorphic to a unique ordinal.*

Proof. Exercise. □

Now thinking of ordinals as numbers we can generalize many classical concepts.

Definition 1.15. For a given ordinal α, an *(α-)sequence* is a function whose domain is α. An *enumeration* of a set X is a sequence whose range is X. Often the notation $\langle a_0, a_1, \ldots, a_\beta, \ldots \rangle$, $\beta < \alpha$, or $(a_\beta)_{\beta<\alpha}$ is used for α-sequences.

We can also do arithmetic with ordinals. Recall that for linear orders $(I, <_I)$ and $(J, <_J)$, their sum $I + J$ denotes the linear order defined by $x < y$ if ($x, y \in I$ and $x <_I y$ or $x, y \in J$ and $x <_J y$ or $x \in I$ and $y \in J$). Then one can define the *sum* of two ordinals as the (unique) ordinal isomorphic to the well order one gets by summing the orders $\alpha + \beta$. One can also define the *product* of two ordinals $\alpha \cdot \beta$ as the unique ordinal isomorphic to the well order $(\alpha \times \beta, <_H)$ where $<_H$ is Hebrew lexicographic order (i.e., $\alpha \cdot \beta$ is 'α, β times'). A detailed treatment of ordinal arithmetic can be found in [End77].

The most common way of using transfinite induction is by using the first transfinite induciton principle but splitting up the successor and limit cases. One then proves (for a given property P)

1. (initial step) $P(0)$ holds.
2. (successor case) If $P(\alpha)$ holds, then $P(\alpha + 1)$ also holds.
3. (limit case) If γ is a limit ordinal and $P(\beta)$ holds for all $\beta < \gamma$, then $P(\gamma)$ holds.

Then one concludes that $P(\alpha)$ holds for all ordinals α.

Transfinite induction helps us prove that given properties hold for all ordinals. A related tool is *transfinite recursion*, that allows us to construct (transfinite) sequences of objects.

Theorem 1.16 (Transfinite recursion)**.** *Let G be a function on $\bigcup_{\beta<\alpha} {}^{\beta}X$ into X. Then there exists a unique α-sequence f such that $f(\beta) = G(f \restriction \beta)$ for all $\beta < \alpha$.*

Proof. We prove the recursion theorem by (transfinite) induction on α. So let α_0 be an ordinal and suppose the claim holds for all $\alpha < \alpha_0$. Let $G : \bigcup_{\beta<\alpha_0} {}^{\beta}X \to X$ be a function.

By the assumption, for each $\gamma < \alpha_0$, there exists a unique function $f_\gamma : \gamma \to X$ such that $f_\gamma(\beta) = G(f_\gamma \restriction \beta)$ for all $\beta < \gamma$.

Now if α_0 is a successor, i.e., $\alpha_0 = \gamma + 1$ for some γ, let

$$f = f_\gamma \cup \{\langle \gamma, G(f_\gamma)\rangle\}.$$

Then $f(\beta) = G(f \restriction \beta)$ for all $\beta < \gamma + 1$.

If α_0 is a limit, then if $\delta < \gamma < \alpha_0$ we have for all $\beta < \delta$:

$$f_\gamma \restriction \delta(\beta) = f_\gamma(\beta) = G(f_\gamma \restriction \beta) = G((f_gamma \restriction \delta) \restriction \beta).$$

By uniqueness of f_δ, we must have $f_\gamma \restriction \delta = f_\delta$, i.e., $f_\delta \subset f_\gamma$. Thus the function $f = \bigcup_{\gamma<\alpha_0} f_\gamma$ is well-defined, and for all $\beta < \alpha_0$,

$$f(\beta) = f_{\beta+1}(\beta) = G(f_{\beta+1} \restriction \beta) = G(f \restriction \beta).$$

For uniqueness, note that if f' is a function from α_0 to X satisfying $f'(\beta) = G(f' \restriction \beta)$ for all $\beta < \alpha_0$, then by the uniqueness of f_β,

$$f' \restriction \beta = f_\beta = f \restriction \beta$$

for all $\beta < \alpha_0$. Thus

$$f'(\beta) = G(f' \restriction \beta) = G(f \restriction \beta) = f(\beta)$$

for all $\beta < \alpha_0$, so $f = f'$. □

Transfinite recursion gives us another possibility of defining ordinal arithmetic, by first recursively defining addition (using the successor function) and then defining multiplication recursively based on addition. The details can be found e.g. in [End77].