Department of Mathematics and Statistics
Riemannian geometry
Exercise 7
22.3.2016

Note: No classes on Wednesday, March 16.

1. Let (M, g) and $(\tilde{M}, \tilde{g})$ be Riemannian manifolds and $\varphi\colon M \to \tilde{M}$ an isometry. Prove that the curvature tensor fields R of M and $\tilde{R}$ of $\tilde{M}$ satisfy an equation
$$\tilde{R}(\varphi_* X, \varphi_* Y)\varphi_* Z = \varphi_*(R(X, Y)Z)$$
for all $X, Y, Z \in \mathcal{T}(M)$. Prove furthermore that the Riemannian curvature tensors R and $\tilde{R}$ satisfy $\varphi^* \tilde{R} = R$.
You may use the fact that $\varphi^* \tilde{\nabla} = \nabla$, that is,
$$\varphi_*(\nabla_X Y) = \tilde{\nabla}_{\varphi_* X}(\varphi_* Y).$$

2. Prove that for all $f, g \in C^\infty(M)$
   (a) $R(fX_1 + gX_2, Y)Z = fR(X_1, Y)Z + gR(X_2, Y)Z$;
   (b) $R(X, fY_1 + gY_2)Z = fR(X, Y_1)Z + gR(X, Y_2)Z$;
   (c) $R(X, Y)(fZ) = fR(X, Y)Z$;
   (d) $R(X, Y)(Z + W) = R(X, Y)Z + R(X, Y)W$.

3. Let M be a Riemannian manifold. Prove that
   (1) $R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0$;
   (2) $\langle R(X, Y)Z, W\rangle = \langle R(Z, W)X, Y\rangle$;
   (3) $\langle R(X, Y)Z, W\rangle = -\langle R(X, Y)W, Z\rangle$.

4. Let $P \subset T_pM$ be a 2-dimensional subspace and let $u, v \in P$ be linearly independent. Prove that $K(u, v)$ is independent of the choice of $u, v \in P$.

5. Show that any connected Riemannian manifold (M, g) admits a Riemannian metric $\tilde{g} = \varphi g$, where $\varphi\colon M \to \mathbb{R}$ is a positive C^∞-function, such that $(M, \tilde{g})$ is complete.