Department of Mathematics and Statistics
Riemannian geometry
Exercise 6
1.3.2016

1. Prove the following version of the Gauss lemma: Let $p \in M$ and $v \in T_pM$ a vector such that $\exp_p v$ is defined. Let $w \in T_v(T_pM) = T_pM$. Then
$$\langle \exp_{p*v}(v), \exp_{p*v}(w) \rangle = \langle v, w \rangle.$$

2. Show that any connected Riemannian manifold (M, g) admits a Riemannian metric $\tilde{g} = \varphi g$, where $\varphi \colon M \to \mathbb{R}$ is a positive C^∞-function, such that $(M, \tilde{g})$ is bounded. In other words, there exists a constant C such that $d_{\tilde{g}}(x, y) \leq C$ for all $x, y \in M$.
   [Hint: The following facts may be useful. (a): If $h \colon M \to \mathbb{R}$ is a non-negative continuous function, then there exists a C^∞-function $f \colon M \to \mathbb{R}$ s.t. $f(x) > h(x)$ for all $x \in M$. (b): For every $\varepsilon > 0$ and for every $p, q \in M$, there exists an admissible path $\gamma \colon [0, L] \to M$ such that $L = \ell(\gamma) \leq d(p, q) + \varepsilon$ and $|\dot{\gamma}_t| = 1$ except for finitely many $t \in [0, L]$.]

3. Let M and N be Riemannian manifolds and $f \colon M \to N$ a diffeomorphism. Suppose that N is complete and that there exists a constant $c > 0$ such that
$$|v| \geq c|f_{*p}v|$$
for all $p \in M$ and for all $v \in T_pM$. Prove that M is complete.

4. Let M be a complete connected Riemannian manifold, N a Riemannian manifold and $f : M \to N$ a smooth mapping that is a local isometry. Suppose that for every $x, y \in N$ there exists a unique geodesic from x to y. Prove that f is bijective (and hence an isometry).
   [You may use the fact that local isometries preserve geodesics.]

5. Let M be a smooth manifold, N a Riemannian manifold and $f \colon M \to N$ a surjective local diffeomorphism. Introduce on M a Riemannian metric such that f is a local isometry. Furthermore, show by examples that M (equipped with the Riemannian metric introduced above) need not be complete even if N is complete.