Department of Mathematics and Statistics
Riemannian geometry
Exercise 11
3.5.2016

The course can be passed by an exam (e.g. May 18 (12:00-16:00) or May 25 (12:00-16:00)).

1. Suppose that M has constant sectional curvature κ. Let $p \in M$ and let $e_1, \ldots, e_n$ be an orthonormal basis of T_pM, (U, φ) the corresponding normal chart at p such that U is a normal ball, and g_{ij} the corresponding component functions of the Riemannian metric. Prove that
$$g_{ij}(\exp_p v) = \frac{v^i v^j}{|v|^2} + \frac{\mathbf{S}_\kappa^2(|v|)}{|v|^2}\left(\delta_{ij} - \frac{v^i v^j}{|v|^2}\right),$$
for $\exp_p v \in U,\ v = v^i e_i \neq 0$. Here
$$\mathbf{S}_\kappa(t) = \begin{cases} \frac{1}{\sqrt{\kappa}} \sin(\sqrt{\kappa}\, t), & \kappa > 0, \\ t, & \kappa = 0, \\ \frac{1}{\sqrt{-\kappa}} \sinh(\sqrt{-\kappa}\, t), & \kappa < 0. \end{cases}$$

2. Let M and $\tilde{M}$ be n-dimensional Riemannian manifolds of constant sectional curvature κ. Prove that for any $p \in M$ and $\tilde{p} \in \tilde{M}$ there exists $\delta > 0$ such that the geodesic balls $B(p, \delta) \subset M$ and $B(\tilde{p}, \delta) \subset \tilde{M}$ are isometric.

3. Suppose that $\gamma^v(t_0)$ is the cut point of $p = \gamma^v(0)$ along γ^v. Prove that at least one of the following conditions holds
   (a) $\gamma^v(t_0)$ is the first conjugate point of p along γ^v, or
   (b) there exists a unit speed geodesic $\sigma \neq \gamma^v|[0, t_0]$ from p to $\gamma^v(t_0)$ such that $\ell(\sigma) = t_0 = \ell\big(\gamma^v|[0, t_0]\big)$.

4. Prove that, for a complete Riemannian manifold, $\mathrm{inj}(p) = d(p, C(p))$ provided $C(p) \neq \emptyset$, where $C(p)$ is as in Definition 8.15.

5. Prove that $p \mapsto \mathrm{inj}(p)$ is a continuous positive function on any Riemannian manifold.