Exercise 1. Consider maps $f = (u, v, w) \in C^\infty(\mathbb{R}^3; \mathbb{R}^3)$ with differential matrix

$$Df(x) = \begin{pmatrix}
u_x & u_y & u_z \\
v_x & v_y & v_z \\
w_x & w_y & w_z \\
\end{pmatrix}$$

Show that the $2 \times 2$ minor $L(Df) := \det \begin{pmatrix}v_y & v_z \\
v_x & w_z \end{pmatrix} = v_yw_z - v_zw_y$ is a null-Lagrangian.

Solution 1. It is sufficiently simple to compute the Euler-Lagrange equations for the expression $L(Df) = v_yw_z - v_zw_y$. We obtain that

$$-\nabla \cdot D_{P1}L(Df) + D_{z1}L(Df) = 0$$
$$-\nabla \cdot D_{P2}L(Df) + D_{z2}L(Df) = -\nabla \cdot (0, w_z, -w_y) + 0 = -w_{yz} + w_{yz} = 0$$
$$-\nabla \cdot D_{P3}L(Df) + D_{z3}L(Df) = -\nabla \cdot (0, -v_z, v_y) + 0 = v_{zy} - v_{yz} = 0$$

There is no dependence of $z$ (the variable in whose place you put the function $f$) in $L(Df)$, so the derivatives $D_{z2}L(Df)$ vanish above. The expression $D_{P1}L(Df)$ denotes a gradient of the function

$$L(Df) = L \begin{pmatrix}u_x & u_y & u_z \\
u_x & v_y & v_z \\
w_x & w_y & w_z \end{pmatrix}$$

with respect to the variables on the row $i$. Since the Euler-Lagrange equations are always satisfied, $L$ is a null Lagrangian.

Exercise 2. [Evans, Problem 8.7.7] Prove that $L(P) := \text{trace}(P^2) - \text{trace}(P)^2$ is a null Lagrangian. Here the trace of an $n \times n$ matrix $A = (a_{ij})_{i,j=1}^n$ is defined $\text{trace}(A) = \sum_{j=1}^n a_{jj}$.

Solution 2. Let us first expand the formula, denoting $P = (p_{ij})$:

$$\text{tr}(P^2) - \text{tr}(P)^2 = \sum_{i,j=1}^n p_{ij}p_{ji} - \left( \sum_{i=1}^n p_{ii} \right)^2 = \sum_{i,j=1}^n p_{ij}p_{ji} - p_{ii}p_{jj}.$$  

The expression $p_{ij}p_{ji} - p_{ii}p_{jj}$ is a $2 \times 2$ subdeterminant of the matrix $\begin{pmatrix}p_{ii} & p_{ij} \\
p_{ji} & p_{jj} \end{pmatrix}$ obtained from the matrix $P$ by removing all rows and columns except $i$ and $j$. It happens that
each of these subdeterminants is a null Lagrangian, much to the same reason as why the expression of Exercise 1 was one (in fact, subdeterminants are always null Lagrangians). To prove this, we compute the Euler-Lagrange equations for $L_{ij} = p_{ij}p_{ji} - p_{ii}p_{jj}$ as

$$-\nabla \cdot D_{pk} L_{ij}(Df) + D_{zy} L_{ij}(Df) = 0,$$
$$\text{when } k \neq i, j$$

$$-\nabla \cdot D_{pz} L_{ij}(Df) + D_{zj} L_{ij}(Df) = -f_{z_j}^j + f_{z_i}^i = 0$$
$$-\nabla \cdot D_{pz} L_{ij}(Df) + D_{zi} L_{ij}(Df) = f_{z_j}^j - f_{z_i}^i = 0.$$

**Exercise 3.** [Evans, Problem 8.7.4] Assume $\eta : \mathbb{R}^n \to \mathbb{R}$ is $C^1$.

(i) Show that $L(P, z, x) := \eta(z) \det P$ is a null Lagrangian; here $P \in M^{n \times n}, z \in \mathbb{R}^n$.

(ii) Deduce that if $f : \mathbb{R}^n \to \mathbb{R}^n$ is $C^1$, then

$$\int_\Omega \eta(f) \det(Df) dx$$

depends only on $f|_{\partial \Omega}$.

**Solution 3.** a) We compute again by Euler-Lagrange equations.

$$-\nabla_x \cdot D_{pk} L(Df, f) + D_{zy} L(Df, f)$$
$$= -\nabla_x \cdot (\eta(f)D_{pk} \det Df) + (D_{zy} \eta(f)) \det Df$$
$$= -\eta(f)\nabla_x \cdot D_{pk} \det Df - \nabla_x \eta(f) \cdot D_{pz} \det Df + (D_{zy} \eta(f)) \det Df$$

The first term is just $\eta(f)$ times the Euler-Lagrange equation for the Jacobian $\det Df$. The Jacobian is known to be a null Lagrangian, so we do not repeat the proof here. One may compute

$$\nabla_x \eta(f) = \left( \sum_{j=1}^n \eta_{zz}^j(f)f^j_i \right)^n_{i=1}$$

We use the cofactor expansion for the determinant with row $k$:

$$\det Df = \sum_{i=1}^n (-1)^{i+k} f^k_i M_{ki},$$

where $M_{ki}$ denotes the determinant of the matrix we get by removing row $k$ and column $i$ from $Df$. Thus

$$D_{pz} \det Df = \left( (-1)^{i+k} M_{ki} \right)_{i=1}^n.$$

This finally gives

$$-\nabla_x \eta(f) \cdot D_{pz} \det Df = -\sum_{i=1}^n \sum_{j=1}^n \eta_{zz}^j(f)f^j_i (-1)^{i+k} M_{ki}.$$
Note that the term in the above sum with \( j = k \) is exactly \((D_x \eta)(f) \det Df\), which cancels out the similar term in the Euler-Lagrange equation. The rest is equal to

\[
- \sum_{j \neq k} \eta_{z_j}(f) \sum_{i=1}^{n} f_i^j (-1)^{i+k} M_{ki}
\]

We now expand each subdeterminant \((\text{cof } Df)_{ki}\) with respect to the \( j \)th row, which gives

\[
(\text{cof } Df)_{ki} = \sum_{l \neq i} (-1)^{l+j} (-1)^{\chi(l>i)+\chi(j>k)} f_i^j f_l^j M_{ki, jl},
\]

where \(M_{ki, jl}\) denotes the determinant of the matrix we get by removing rows \(k\) and \(j\) and columns \(i\) and \(l\) from \(Df\). Here also \(\chi(a > b)\) is equal to 1 if \(a > b\) and 0 otherwise. The factor \((-1)^{\chi(l>i)+\chi(j>k)}\) comes from the fact that when we remove row \(k\) and column \(i\), we have to swap all the \(\pm\)-signs that come after. Thus what remains of the Euler-Lagrange equation reads

\[
- \sum_{j \neq k} \eta_{z_j}(f) \sum_{i=1}^{n} \sum_{l \neq i} (-1)^{i+k+l+j} (-1)^{\chi(l>i)+\chi(j>k)} f_i^j f_l^j M_{ki, jl}.
\]

Obviously \(M_{ki, jl} = M_{kl, ji}\). But this means that in the last two sums, the terms \((i, l)\) and \((l, i)\) cancel each other out because of the factor \((-1)^{\chi(l>i)}\). This shows that the whole expression is zero, and hence that \(L\) is a null Lagrangian.

b) Follows from the alternate characterization of null Lagrangians, and the fact that the above computation may be generalized to \(f \in C^1\) in the weak sense.

**Exercise 4.** [Evans, Problem 8.7.5] If \(f : \mathbb{R}^n \to \mathbb{R}^n\) is as in Problem 3, fix \(x_0 \notin f(\partial \Omega)\). If \(r\) is so small that \(B(x_0, r) \cap f(\partial \Omega) = \emptyset\), choose a \(C^1\)-map \(\eta\) so that \(\int_{\mathbb{R}^n} \eta(z)dz = 1\) and \(\eta(x) = 0\) when \(|x - x_0| \geq r\).

Define

\[
\deg(f, x_0) = \int_{\Omega} \eta(f) \det(Df)dx,
\]

the *degree* of \(f\) relative to \(x_0\). Prove that the degree is an integer.

**Solution 4.** Solution will be added a bit later.

**Exercise 5.** In geometric function theory one studies the *distortion* of a map \(f : \mathbb{R}^2 \to \mathbb{R}^2\). Writing \(f = (u, v)\) and assuming that the Jacobian \(\det(Df(x)) > 0\) is positive almost everywhere, the distortion is defined by

\[
K(f) := \frac{|\partial_x u|^2 + |\partial_y u|^2 + |\partial_x v|^2 + |\partial_y v|^2}{\det(Df)}
\]
Show that the functional \( L(Df) := K(f) \) is polyconvex; do this by first showing that \( F(x, y) = x^2/y \) is convex on \((0, \infty) \times (0, \infty)\).

[Hint: You need to show that \( F(x, y) - F(a, b) \geq 2ab^{-1}(x - a) - ab^{-2}(y - b) \)]

**Note.** In higher dimensions the distortion of a map \( f : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is defined by

\[
K(f) := \left[ \sum_{j,k=1}^{n} |\partial x_j f^k|^2 \right]^{n/2} / \det(Df)
\]

so that \( K(tf) = K(f) \) for all \( t \in \mathbb{R} \). Also in higher dimensions the distortion is polyconvex, but the algebra to prove this is a little more difficult.

**Solution 5.** Let us first show that \( F(x, y) = x^2/y \) is convex as a function of two real variables. Let \( 0 < t < 1 \). We want to prove that

\[
F(tx + (1 - t)a, ty + (1 - t)b) \leq tF(x, y) + (1 - t)F(a, b)
\]

This reduces to

\[
\Leftrightarrow \frac{t^2x^2 + 2t(1 - t)ax + (1 - t)^2a^2}{ty + (1 - t)b} \leq \frac{tx^2}{y} + \frac{(1 - t)a^2}{b}
\]

\[
\Leftrightarrow t^2x^2yb + 2t(1 - t)axyb + (1 - t)^2a^2y \leq (ty + (1 - t)b)(tx^2b + (1 - t)a^2y)
\]

\[
\Leftrightarrow 2t(1 - t)axyb \leq t(1 - t)(x^2l^2 + a^2y^2)
\]

\[
\Leftrightarrow 0 \leq t(1 - t)(xb - ay)^2.
\]

Thus our expression is convex. Now let us consider the distortion as a function

\[
K(P, r) = \frac{p_{11}^2 + p_{12}^2 + p_{21}^2 + p_{22}^2}{r} = \frac{|P|^2}{r}.
\]

Here \( |P| = (p_{11}^2 + p_{12}^2 + p_{21}^2 + p_{22}^2)^{1/2} \), and we remark that \( |P|^2 \) is a convex function of the matrix \( P \) because the function \( f(x) = x^2 \) is convex as well. Then if \( 0 < t < 1 \),

\[
K(tp_1 + (1 - t)p_2, tr_1 + (1 - t)r_2) = \frac{|tp_1 + (1 - t)p_2|^2}{tr_1 + (1 - t)r_2}
\]

\[
\leq \frac{t|P_1|^2 + (1 - t)|P_2|^2}{tr_1 + (1 - t)r_2}
\]

\[
\leq \frac{t|P_1|^2}{r_1} + \frac{(1 - t)|P_2|^2}{r_2}
\]

\[
= tK(p_1, r_1) + (1 - t)K(P_2, r_2).
\]

This proves the polyconvexity.