**Exercise 1.** Recall the continuous linear operators $T : X \to Y$ between Banach spaces $X$ and $Y$; and that these have the norm $\|T\| := \sup\{\|Tx\| : \|x\| \leq 1\}$. If $T_k : X \to Y$ are compact linear operators and $\|T - T_k\| \to 0$, show that $T : X \to Y$ is compact.

[Hint: Recall the different characterisations of compactness in Banach spaces]

**Solution 1.** Recall the following notion of precompactness: A set is precompact iff for every $\epsilon > 0$ it admits a finite cover of balls with radius $\epsilon$. We prove now that $T(B)$ is precompact, where $B$ is the unit ball in $X$. Let $\epsilon > 0$. Take $m$ so large that $\|T - T_m\| < \epsilon/2$, and choose by compactness a finite cover of $T_m(B)$ with balls of radius $\epsilon/2$. Let the centers of these balls be $y_1, \ldots, y_M$. It is enough to prove that the balls $B(y_1, \epsilon), \ldots, B(y_M, \epsilon)$ cover $T(B)$. Let $x \in B$. Then $T_m(x) \in T_m(y_j, \epsilon/2)$ for some $j$. Thus

$$\|T(x) - y_j\|_Y \leq \|T(x) - T_m(x)\|_Y + \|T_m(x) - y_j\|_Y < \epsilon/2 + \epsilon/2 = \epsilon.$$ 

This proves the claim.

**Exercise 2.** Let $B = B(0,1) \subset \mathbb{R}^2$. Then, as will be discussed later,

$$u(x) := (Tf)(x) = \int_B \log |x - y|f(y)dy$$

is a solution to the Poisson equation $\Delta u = f$. Show that for $2 < p < \infty$, $T : L^p(B) \to W^{1,p}(B)$ is a continuous linear operator. Deduce that $T\|_{L^p(B)}$ is compact as an operator $T : L^p(B) \to L^p(B)$.

**Solution 2.** Will be added later.

**Exercise 3.** Suppose $u \in W^{1,p}(\Omega)$, for some $1 < p < \infty$. If $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz-continuous with $f(0) = 0$, use difference quotients to show that $f \circ u \in W^{1,p}(\Omega)$.

This is a (strong !) generalisation of Problem 4/Exercises 2. As an application, show that the positive part $u^+ \in W^{1,p}(\Omega)$; here $u^+(x) = u(x)$ if $u(x) \geq 0$ and $u^+(x) = 0$ otherwise.

**Solution 3.** We use Theorem 3 from Evans’ book, section 5.8. Suppose that $u \in W^{1,p}(\Omega)$. Then by Evans’ theorem, we have the following bound for the difference quotients

$$||D_hu||_{L^p(V)} \leq C||Du||_{L^p(\Omega)},$$
where one can check that the constant $C$ does not depend on the compact subset $V \subset \Omega$. If $L$ is the Lipschitz constant of $f$, then we may now estimate that
\[
\left| \frac{f(u(x + he_j)) - f(u(x))}{h} \right| \leq L \left| \frac{u(x + he_j) - u(x)}{h} \right|,
\]
and thus
\[
\left\| D_h(f \circ u) \right\|_{L^p(V)} \leq L \left\| D_h u \right\|_{L^p(V)} \leq C_1 \left\| D u \right\|_{L^p(\Omega)}.
\]
Letting $V \to \Omega$ gives that $D(f \circ u) \in L^p(\Omega)$. The estimate
\[
\left| f(u(x)) \right| = \left| f(u(x)) - f(0) \right| \leq L \left| u(x) \right|
\]
also gives that $f \circ u \in L^p(\Omega)$. Thus $f \circ u \in W^{1,p}(\Omega)$. Applying this result to the Lipschitz function $f(x) = \max(x, 0)$ proves the second part of the exercise too.

**Exercise 4.** Suppose $1 < s \leq p < \infty$ and $|\Omega| < \infty$, so that $L^p(\Omega) \subset L^s(\Omega)$. If $\|f_k\|_{L^p(\Omega)} \leq 1$, $k = 1, 2, \ldots$ and if $f_k \to f$ weakly in $L^s(\Omega)$, show that
\[
f \in L^p(\Omega) \quad \text{and} \quad \|f\|_{L^p(\Omega)} \leq 1.
\]

**Solution 4.** We denote the Hölder-conjugates of $p$ and $s$ by $p'$ and $s'$ respectively. Recall from the duality of $L^p$ spaces that
\[
\left\| f \right\|_{L^p} = \sup \left\{ \left| \int_{\Omega} f g \, dx \right| : \left\| g \right\|_{L^{p'}} \leq 1 \right\}.
\]
Let $g \in L^{s'}$ be such that $\|g\|_{L^{p'}} \leq 1$. Then by weak convergence,
\[
\left| \int_{\Omega} f g \, dx \right| = \lim_{k \to \infty} \left| \int_{\Omega} f_k g \, dx \right| \leq \|f_k\|_p \leq 1.
\]
Now since $s \leq p$, we have $s' \geq p'$ and thus $L^{s'} \subset L^{p'}$. The above inequality proves that
\[
\sup \left\{ \left| \int_{\Omega} f g \, dx \right| : g \in L^{s'}, \|g\|_{L^{p'}} \leq 1 \right\} \leq 1.
\]
However, $L^{s'}$ is dense in $L^{p'}$, which shows that we must also have
\[
\sup \left\{ \left| \int_{\Omega} f g \, dx \right| : \|g\|_{L^{p'}} \leq 1 \right\} \leq 1.
\]
This concludes the proof that $\left\| f \right\|_{L^p} \leq 1$. 

2
Exercise 5. (Evans, problem 5.10.11) Recall the difference quotients $D^h_j u(x)$ and the difference gradient $D^h u(x) = (D^h_1 u(x), D^h_2 u(x), \ldots, D^h_n u(x))$.

Prove that Theorem 3 in Evans/Section 5.8 does not hold at $p = 1$: That is, show by an example that if we have $\|D^h u\|_{L^1(\Omega')} \leq C$ for all $\Omega' \subset \subset \Omega$ and for all $|h| \leq \text{dist}(\Omega', \partial \Omega)$, it does not necessarily hold that $u \in W^{1,1}(\Omega)$.

Solution 5. Note that the statement of this Exercise differs quite a bit from the actual Problem 11 in Evans, as Evans doesn’t require the “for all $\Omega' \subset \subset \Omega$”. Nevertheless, our counterexample will be local so it solves both questions.

For the counterexample, choose $u(x)$ as the characteristic function of some set $V \subset \subset \Omega$. It’s enough if the set $V$ has $C^1$-boundary, so a ball for example. Then $D_h u$ will be bounded in the $L^1$-norm uniformly in $h$. This is because the difference quotient

$$\frac{u(x + he_j) - u(x)}{h}$$

may only attain the values $\pm 1/h$ and 0. The set in which it attains the values $\pm 1/h$ is contained in the set

$$\{x \in \Omega : \text{dist}(x, \partial V) \leq h\}.$$ 

The above set has measure at most $Ch$ for some constant $C$. Thus

$$\|D_h u\|_{L^1(\Omega)} \leq \int_{\text{dist}(x, \partial V) \leq h} \frac{1}{h} dx \leq C.$$ 

However, $u$ is not in $W^{1,1}(\Omega)$ even locally since it doesn’t have proper weak derivatives.