Some additional details/material covered in the lectures

1. If $n = 1$, every $u \in W^{1,p}(\Omega)$ has abs. cont. representative.

2. Higher order reflection.

3. Partitions of unity.

4. Gagliardo-Nirenberg-Sobolev inequality

5. Money's inequality

6. On compact embeddings

7. Remark on compact embeddings of $W^{1,p}(\Omega)$

8. On weak convergence and weak compactness

9. A lemma on weak limits in $L^p(\Omega)$


12. Weierstrass example where no minimizer exists.
Further results from Functional Analysis and weak convergence

Return to Dirichlet Principle
Proposition 1. If $v = 1$, $1 \leq p < \infty$ and $-\infty < a < b < \infty$ then $u \in W^{1,p}(a,b) \iff u(x) = f(x)$ for a.e. $x \in (a,b)$, where $f$ is absolutely continuous and $f' \in L^p(a,b)$.

In particular, $u(x) - u(y) = \int_y^x u'(t) \, dt$ for a.e. $x, y \in [a,b]$.

Remark: above $f'$ is the pointwise derivative that exists for absolutely continuous functions at a.e. $x \in (a,b)$.

Proof: "$\Rightarrow\"$ If $Du$ denotes the weak derivative of $u$, let 

$$ q(x) := \int_a^x Du(t) \, dt $$

Since $Du \in L^p(a,b) \subset L^1(a,b)$, $f$ is absolutely continuous [see Real Analysis I, notes p. 58], and so $f'(x)$ exist for a.e. $x$.

Claim: $f'$ is also the weak derivative of $f$.

Indeed, if $\phi \in C^\infty_0 (a,b) \Rightarrow \phi f$ is abs. continuous, and

so [by Real Analysis I, p. 58] \Rightarrow

$$ 0 = (\phi f)(b) - (\phi f)(a) = \int_a^b (\phi f)'(t) \, dt = \int_a^b \phi'(t) f(t) \, dt + \int_a^b \phi(t) f'(t) \, dt. $$

$$ \therefore \int_a^b \phi' \cdot u' \, dt = -\int_a^b \phi' f(t) \, dt \Rightarrow f' \text{ the weak derivative} $$

But $\circ$ and Real Anal I, p. 58 \Rightarrow $f'(x) = Du(x)$ for a.e. $x$.

Thus the weak derivative of $u - f$, $D(u - f) = 0$, and $u - f$ is constant a.e. Finally, $u = f + C$ is abs. continuous. \hfill $\blacksquare$ (Exercise 4)
Lemma (higher order reflection). Let \( B = B(0,1) \subset \mathbb{R}^n \) and
\[
B_+ = B \cap \{ x_n \geq 0 \}.
\]
If \( u \in C^\infty(B_+) \), define its extension to \( B \) by
\[
\overline{u}(x) = \begin{cases} 
  u(x), & x \in B_+ \\
  \sum_{j=1}^{n} a_j u(x_1, \ldots, x_{n-1}, -\beta_j x_n), & x \in B \setminus B_+.
\end{cases}
\]
If \( \beta_j > 0 \) and
\[
\alpha_1 \beta_1^k + \alpha_2 \beta_2^k + \cdots + \alpha_{k+1} \beta_{k+1}^k = (-1)^l, \quad l = 0, \ldots, k,
\]
then \( u \in C^k(B) \).

Example: Evans, (13) p. 255, takes \( k = 1 \), \( x = -3 \), \( x_2 = 4 \), \( \beta_1 = 1 \), \( \beta_2 = 1 \).

Proof of lemma: Only continuity of \( \overline{u} \) and \( D^{\alpha}_{x_m} \overline{u} \) on \( \{ x_n = 0 \} \) needs to be checked.

1. \( \lim_{x_n \to 0^+} \overline{u}(x_1, \ldots, x_{n-1}, x_n) = \lim_{x_n \to 0^-} \overline{u}(x_1, \ldots, x_{n-1}, x_n) \)
\[
\iff \alpha_1 + \alpha_2 + \cdots + \alpha_{k+1} = 1
\]

2. \( \lim_{x_n \to 0^+} \frac{\partial}{\partial x_m} \overline{u}(x_1, \ldots, x_{n-1}, x_n) = \lim_{x_n \to 0^-} \frac{\partial}{\partial x_m} \overline{u}(x_1, \ldots, x_{n-1}, x_n) \)
\[
\iff \alpha_1 (-\beta_1) + \alpha_2 (-\beta_2) + \cdots + \alpha_{k+1} (-\beta_{k+1}) = 1
\]

as seen by derivation of the expressions of \( \overline{u} \).

3. Continuity of \( D^{\alpha}_{x_m} \overline{u} \iff x_1 (\beta_1)^l + \cdots + x_{k+1} (\beta_{k+1})^l = 1 \)
follows similarly. Same holds for \( D^{\alpha}_{x_m} \overline{u} = \frac{\partial}{\partial x_m} \left( \frac{\partial}{\partial x_1} \cdots \frac{\partial}{\partial x_{n-1}} \overline{u} \right) \).
Remark (a) Since extension $\overline{u}$ is obtained
some of linear coordinate changes, we have

$\| \overline{u} \|_{W^{k,p}(B^+)} \leq C_{k,p} \| u \|_{W^{k,p}(\Omega)}$.

b) $u \mapsto \overline{u}$ is linear.

c) With above higher order reflection the extension theorem (S.4. Theorem 5) from Evans generalizes to $W^{k,p}$-spaces, with same proof:

**Theorem** Suppose $\Omega \subset \mathbb{R}^n$ is bounded domain with $C^\infty$-boundary \cite{Evans}. Suppose $\Omega \subset \subset V \subset \mathbb{R}^n$. Then there is a bounded linear operator, called extension operator,

$$E : W^{k,p}(\Omega) \to W^{k,p}(\mathbb{R}^n),$$

such that for each $u \in W^{k,p}(\Omega)$ we have

(i) $Eu = u$ a.e. in $\Omega$

(ii) $\text{supp}(Eu) \subset \subset V$

(iii) $\| Eu \|_{W^{k,p}(\Omega)} \leq C \| u \|_{W^{k,p}(\mathbb{R}^n)}$

where the constant $C$ depends only on $k, p, \Omega$ and $V$. 
Partition of unity

Lemma
If \( V \subset \mathcal{N} \) \( \left[ \text{i.e. } \overline{V} \subset \mathcal{N} \text{ compact} \right] \), then
\[ \exists \ \phi \in C_0^\infty (\mathcal{N}) \] with \( \phi(x) = 1 \ \forall \ x \in V \), \( 0 \leq \phi \leq 1 \ \forall \ x \in \mathcal{N} \).

Proof:
\[ f_\varepsilon (x) = \frac{1}{\varepsilon^n} \int_\mathbb{R}^n \eta (\frac{x-y}{\varepsilon}) \, dy \]
with \( \int_\mathbb{R}^n \eta (y) \, dy = 1 \) and \( \text{supp}(\eta) \subset B(0,1) \)
write \( V^\varepsilon := \{ x \in \mathbb{R}^n : \text{dist} (x,V) < \varepsilon \} \) and choose \( \varepsilon > 0 \) so small that \( V^\varepsilon \subset \subset \mathcal{N} \).

If \( 0 < \varepsilon < 5 \), set \( \phi(x) = (f_\varepsilon * \chi_{V^\varepsilon})(x) \)

Since \( \text{supp}(\phi) \subset V^{\varepsilon + \varepsilon} \subset \subset \mathcal{N} \), \( \phi \in C_0^\infty (\mathcal{N}) \).

Also \( 0 \leq \phi (x) = \int_{V^\varepsilon} f_\varepsilon (x-y) \, dy \leq \int_{\mathbb{R}^n} f_\varepsilon \, dy = 1 \)
and \( x \in V \Rightarrow f_\varepsilon (x-y) = 0 \) whenever \( \text{dist} (y,V) \geq 5 > \varepsilon \).

Thus \( \phi(x) = \int_{\mathbb{R}^n} \eta (y) \, dy = 1 \ \forall \ x \in V. \quad \Box \)

Proposition
If \( \mathcal{N} = \bigcup_{j=1}^N V_j \), bounded, then one can find \( \phi_j \in C_0^\infty (V_j) \), \( 0 \leq \phi_j \leq 1 \) \( \forall j = 1, \ldots, N \),

such that
\[ \sum_{j=1}^N \phi_j (x) = 1 \ \forall \ x \in \mathcal{N}. \]

Call \( \{ \phi_j \}^N_{j=1} \) a partition of unity in \( \mathcal{N} \), subordinate to covering \( \{ V_j \}_{j=1}^N. \)
Proof of Proposition:

Choose first $\phi \in C_c^\infty \left( \bigcup_{i=1}^N V_i \right)$ such that

$0 \leq \phi \leq 1$ and $\phi(x) = 1$ for $x \in \overline{V}$.

(see lemma on previous page, p. 3a)

Next choose compact sets $K_j \subset V_j$, $j = 1, \ldots, N$, such that $\text{supp}(\phi) \subset \bigcup_{j=1}^N K_j$.

Every point $x \in \text{supp}(\phi)$ has closed neighborhoods contained in some $V_j$; by compactness, any cover $\text{supp}(\phi)$ by finitely many interiors of such neighborhoods can be collected to $K_j$.

Now find $\psi_j \in C_c^\infty (V_j)$ with $0 \leq \psi_j \leq 1$ and $\psi_j \big|_{K_j} = 1$.

(by lemma)

Now find $\psi_j \in C_c^\infty (V_j)$ with $0 \leq \psi_j \leq 1$.

Finally, set

$\phi_1 = \phi \psi_1$, $\phi_2 = \phi \psi_2 (1-\psi_1)$, \ldots, $\phi_j = \phi \psi_j (1-\psi_1) \cdots (1-\psi_{j-1})$

Then $0 \leq \phi_j \leq 1$, $\phi_j \in C_c^\infty (V_j)$ and

$$\sum_{j=1}^N \phi_j - \phi \equiv - \phi \sum_{j=1}^N (1-\psi_j) = 0, \quad x \in \mathbb{R}^m$$

since at every point either $\phi = 0$ or some $(1-\psi_j) = 0$.

As $\phi \big|_{\overline{V}}$, we are done. \( \Box \)
Below is a version of the Gagliardo-Nirenberg-Sobolev inequality, slightly different from Evans' Theorem 1/Section 5.6.1.

**Theorem (Gagliardo-Nirenberg-Sobolev Inequality)**

Let $1 \leq p < n$ and $p^* = \frac{np}{n-p}$. Then for all $u \in W^{1,p}(\mathbb{R}^n)$ we have

\[ \| u \|_{L^{p^*}(\mathbb{R}^n)} \leq C(n,p) \| Du \|_{L^p(\mathbb{R}^n)} \]

**Proof**: Evans proves this for $u \in C^1_c(\mathbb{R}^n)$ on pp. 263-265. Thus we need to extend this from $C^1_c(\mathbb{R}^n)$ to all $u \in W^{1,p}(\mathbb{R}^n)$.

For this, if $u \in W^{1,p}(\mathbb{R}^n)$, \[ u \in W^{1,p}_0(\mathbb{R}^n) \]

Thus have $u_j \in C^\infty_0(\mathbb{R}^n)$ with $\| u_j - u \|_{W^{1,p}(\mathbb{R}^n)} \to 0$

In particular, $\| u_j - u \|_{L^p(\mathbb{R}^n)} \to 0$ as $j \to \infty$.

For a subsequence, $u_j(x) \to u(x)$ a.e. $x \in \mathbb{R}^n$.

But $\sum_{j=1}^{\infty} u_j$ is Cauchy in $L^{p^*}(\mathbb{R}^n)$:

\[ \sum_{j=1}^{\infty} \| u_j - u_j^m \|_{L^{p^*}(\mathbb{R}^n)} \leq C(n,p) \| Du_j - Du_j^m \|_{L^p(\mathbb{R}^n)} \to 0 \]

Therefore have $v \in L^{p^*}(\mathbb{R}^n)$ a.e. $u_j \to v$ in $L^{p^*}(\mathbb{R}^n)$

Again, have subsequence $u_j(x) \to v(x)$ a.e. $x \in \mathbb{R}^n$.

Thus $v = u$ a.e. and so $u \in L^{p^*}(\mathbb{R}^n)$.
But now, \( \| u \|_{L^p} \leq \| u - u_e \|_{L^p(R^n)} + \| u_e \|_{L^p(R^n)} \leq C \),

\[ E_{x^e} + C(n, p) \| Du_{x^e} \|_{L^p(R^n)} \to \quad C(n, p) \| Du \|_{L^p(R^n)} \] (L-\text{Hölder})

\[ (x_{x^e} + C(n, p)) \]

**Corollary** (Sobolev Embedding) \( W^{1,p}(R^n) \subset L^q(R^n) \).

**Corollary** (Evans Theorem 2/Section 5.6.1) / \( \mathcal{D} \subset R^n \) bounded domain with \( C^1 \)-boundary and \( 1 \leq p < n \), then

\[ \| u \|_{L^p(R^n)} \leq C \| u \|_{W^{1,p}(R^n)} = C \| u \|_{L^p(R^n)} \]

\[ \forall u \in W^{1,p}(R^n). \]

**Proof:** \( \mathcal{D} \) admits extension operator \( E : W^{1,p}(R^n) \to W^{1,p}(R^n) \)

(Evans Section 5.4). Thus \( \| u \|_{L^p(R^n)} = \| E u \|_{L^p(R^n)} \leq C \| E u \|_{L^p} \)

\[ \leq C \| u \|_{W^{1,p}(R^n)} \leq C \| u \|_{W^{1,p}(R^n)} \text{ (Sec. 5.4)} \]

**Corollary** (Evans Theorem 2/Section 5.6.1)

If \( \mathcal{D} \subset R^n \) any bounded domain and \( u \in W^{1,p}(\mathcal{D}) \), \( 1 \leq p < n \), then

\[ \| u \|_{L^q(\mathcal{D})} \leq C \| Du \|_{L^p(\mathcal{D})}, \quad 1 \leq q \leq p^* \]

**Proof:** H"{o}lder's)

\[ |u|_{L^q(\mathcal{D})} \leq C \| u \|_{L^p(\mathcal{D})}. \]

Since \( C^0(\mathcal{D}) \) dense in \( W^{1,p}(\mathcal{D}) \), same proof as for Theorem/\( p \). \( \square \) gives the claim.
Money's inequality

Here is a little different approach to Money's inequality, Theorems 4 & 5 in Evans, Section 5.6.2.

Theorem (Money's inequality) If $u \in W^{1,p}(\mathbb{R}^n)$ where $1 \leq p < \infty$, then

$$|u(x) - u(y)| \leq C(n, p) |x - y|^{1 - \frac{n}{p}} \|Du\|_{L^p(\mathbb{R}^n)}$$

for a.e. $x, y \in \mathbb{R}^n$.

Note: Since in general a Sobolev function is defined only almost everywhere, that is best one can say in above inequality! But the result implies that for $p > n$, the Sobolev space has a (Hölder-) continuous representative!!

**Proof of Theorem:**

1. Assume $u \in C^1(\mathbb{R}^n) \cap W^{1,p}(\mathbb{R}^n)$.

If $x, y \in B(x_0, r)$, then

$$u(x) - u(y) = \int_0^1 Du(tx + (1-t)y) \cdot (x-y) \, dt$$

Write:

$$u_{B(x_0, r)} = \frac{1}{|B(x_0, r)|} \int_{B(x_0, r)} u(x) \, dx = \int_{B(x_0, r)} u(x) \, dx$$

Thus

$$|u_{B(x_0, r)} - u_{B(x_0, r)}| = \frac{1}{|B(x_0, r)|} \int_{B(x_0, r)} \left| Du(tx + (1-t)y) \cdot (x-y) \right| \, dt \, dx$$
\[ \begin{align*}
\| u(x) - u(y) \|_p & \leq \| u(x) - u_B(x_0, \nu) \|_p + \| u_B(x_0, \nu) - u(x) \|_p \\
& \leq C(n, p) \| u \|_{L^p(B(x_0, \nu))} \| x - y \|^{1 - m/p} \\
& \leq C(n, p) \| x - y \|^{1 - m/p} \| u \|_{L^p(\mathbb{R}^n)} \\
& \quad \quad \quad \forall x, y \in B(x_0, \nu) 
\end{align*} \]
Let \( u \in W^{1,p}(\mathbb{R}^n) \), generalizing let \( u_\varepsilon = 2\varepsilon \ast u \) be a standard mollification. Then \( u \) is bounded.

\[
|u_\varepsilon(x) - u_\varepsilon(y)| \leq C(n, p) \|x - y\|^1 - \frac{m}{p} \|Du_\varepsilon\|_{L^p(\mathbb{R}^n)}
\]

But Exercises 4 \( \Rightarrow \) \( u_\varepsilon(x) \to u(x) \) at Lebesgue points of \( u \)!

Since by standard \( L^p \)-theory \( Du_\varepsilon = 2\varepsilon \ast Du \to Du \) in \( L^p \)-norm,

\[
\|Du_\varepsilon\|_{L^p} \to \|Du\|_{L^p}.
\]

Thus taking limit \( \varepsilon \to 0 \) get

\[
|u(x) - u(y)| \leq C(n, p) \|x - y\|^1 - \frac{m}{p} \|Du\|_{L^p(\mathbb{R}^n)} \\
\text{at Lebesgue points } x, y.
\]

Since a.e. point is a Lebesgue point (Real analysis I)

the claim follows. \( \square \)

Remark \( \heartsuit \) If \( x \) is a Lebesgue point of \( u \), let \( x \in B(x_0, 1) \) \( \Rightarrow \)

\[
|u(x)| \leq |u(x) - u_B(x_0, 1)| + |u_B(x_0, 1)|
\]

\[
= \frac{1}{|B(x_0, 1)|} \int_{B(x_0, 1)} |u(x) - u_B(x_0, 1)| \, dx + \frac{1}{|B(x_0, 1)|} \int_{B(x_0, 1)} |u_B(x_0, 1)| \, dx
\]

\[
\leq C \|Du\|_{L^p(\mathbb{R}^n)} + C_m \|u\|_{L^p(\mathbb{R}^n)} = C(n, p) \|u\|_{W^{1,p}(\mathbb{R}^n)}
\]

Thus \( \|u\|_{L^\infty(\mathbb{R}^n)} \leq C(n, p) \|u\|_{W^{1,p}(\mathbb{R}^n)} \), \( m < p < \infty \).

By above, each \( u \in W^{1,p}(\mathbb{R}^n) \) has a continuous rep. \( \bar{u} \) with

\[
\|\bar{u}\|_{C^{1,\varepsilon}(\mathbb{R}^n)} := \|\bar{u}\|_{L^\infty} + \sup_{x, y \in \mathbb{R}^n} \frac{|\bar{u}(x) - \bar{u}(y)|}{|x - y|^{1 - \frac{m}{p}}} \leq C \|u\|_{W^{1,p}(\mathbb{R}^n)}
\]
On compact embeddings

0.0. Remarks on embeddings of Banach spaces

Ex. Recall e.g. the consequence of Morrey's inequality:
\[ \|u\|_{C^0,\frac{\alpha}{p}(\mathbb{R}^n)} \leq C_{m,p} \|u\|_{W^{1,p}(\mathbb{R}^n)}, \quad u \in W^{1,p}(\mathbb{R}^n) \]

This says, besides that \( W^{1,p} \subset C^{0,\frac{\alpha}{p}} \), when \( m < p < \infty \),
also little more; it also says, or can be interpreted as saying, that the identity operator

\[ \text{Id}: W^{1,p}(\mathbb{R}^n) \to C^{0,\frac{\alpha}{p}}(\mathbb{R}^n) \]

is a continuous linear operator between these Banach spaces! This point of view is often useful, and
we say that \( W^{1,p}(\mathbb{R}^n) \) is \underline{continuously embedded} to \( C^{0,\frac{\alpha}{p}}(\mathbb{R}^n) \).

Question: Can we say more of the operator?
On compactness

Recall: A metric space \((X, d)\) is precompact if for all \( \epsilon > 0 \), \(X\) can be covered by finitely many sets of diameter \( \leq \epsilon \):
\[ X = \bigcup_{j=1}^{\infty} A_j, \quad \text{diam}(A_j) \leq \epsilon. \]

Also, \(X\) is compact if every open cover of \(X\) has a finite subcover.

A.1. Proportion. If \((X, d)\) metric space, following are equivalent.

(i) \(X\) is compact

(ii) Every sequence \((x_n)\) \(\subseteq X\) has a converging subsequence \(\Rightarrow \) \(X\) sequentially compact \(\Leftrightarrow\)

(iii) \(X\) is precompact and complete.

[For a proof, see Väisälä's book Topology I]

Note: A set \(A \subseteq \mathbb{R}^n\) is compact \(\Leftrightarrow\) A closed and bounded.

This fact is not true in \(\infty\)-dimensional Banach spaces.
A.2. Example

If \( 0 \leq \gamma \in C_c^\infty(\mathbb{R}^n) \) \& \( \text{supp}(\gamma) \subset B(0,1) \),

set

\[ \gamma_l(x) = \gamma(x - 4lx_0), \quad l = 0, 1, 2, \ldots \]

Then \( \text{supp}(\gamma_l) \subset B(4lx_0, 1) \) so that

\[ \text{supp}(\gamma_l) \cap \text{supp}(\gamma_j) = \emptyset \quad \text{for} \quad l \neq j. \]

Thus

\[ \|| \gamma_l - \gamma_j ||_p^p \leq 5 \int_{\mathbb{R}^n} | \gamma_l(x) + \gamma_j(x) |^p dx = 2M^p > 0, \]

where \( M = \| \gamma \|_{L^p(\mathbb{R}^n)} > 0. \)

Hence \( \{ \gamma_l \}_{l=1}^{\infty} \) has no converging subsequences in \( L^p(\mathbb{R}^n) \).

On the other hand, \( \| \gamma_l \|_{L^p} = \| \gamma \|_{L^p} = M \) \( \forall l \), \( \gamma_l \in \{ \gamma \}_{l=1}^{\infty} \subset L^p(\mathbb{R}, M) \)

is bounded.

A very useful criterion for compactness in the space \( C(\overline{\Omega}) = \{ f(\overline{x}, l) \to C \text{ continuous & bounded} \}

equipped with norm \( || f ||_{L^\infty} = \sup_{x \in \overline{\Omega}} | f(x) | \)

is given by the Ascoli-Arzelà theorem:
A.3. Theorem (Ascoli–Arzelà). If \((X, d)\) is a compact metric space and \(H \subset C(X)\), then \(H\) is relatively compact \((i.e., \overline{H}\) compact in \(C(X)\)) if \(H\) is equicontinuous and pointwise bounded.

Here:
- \(H\) equicontinuous at \(x_0 \in X\), if \(\forall \epsilon > 0\), one can find \(\delta > 0\) s.t.,
  \[d(x, x_0) < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon \quad \forall y \in H.\]
- \(H\) equicontinuous, if it is equicontinuous at every point \(x_0 \in X\).
- \(H\) pointwise bounded if \(\forall x \in X\), the set \(\{f(x) : f \in H\}\) is bounded in \(C\).

A.4. Example: Money \(C^m\) \[\|u(x) - u(y)\| \leq c \|x - y\|^{1 - m/p}\]

Thus \(H = \{u \in W^{m,p}(\mathbb{R}^n) : \|u\|_{W^{m,p}(\mathbb{R}^n)} = 1\}\) is equicontinuous in \(\mathbb{R}^n\)!

We also saw on page 5c that \(\|u\|_{L^\infty} \leq c \|u\|_{W^{m,p}(\mathbb{R}^n)}\) \(\Rightarrow\) Thus \(H\) pointwise bounded in \(\mathbb{R}^n\)!
However, \( H \) is not relatively compact in \( C(\mathbb{R}^n) \).

(See A.2 / p. 6c which also in \( C(\mathbb{R}^n) \),

\[ \forall k \neq j, \quad \| 2x_k - 2x_j \|_\infty = \sup_{x \in \mathbb{R}^n} | 2(x_k - x_j, x_j) - 2(x_k - x_j, x_k) | = 2 \| x \|_\infty > 0 \]

But if \( B = B(0, \mathbb{R}) \),

\[ H_{/B} := \{ u \in H \} = \{ u \in H \} \quad \text{such that} \quad \| u \|_{L^1(\mathbb{R}^n)} \leq 1 \]

then by Ascoli–Arzelà \( H_{/B} \) is relatively compact in \( C(0) \).

**Remark.** If \( \Omega \subset \mathbb{R}^n \) is a domain with \( C^1 \)-boundary, we can use the extension operator \( E: W^{1, p}(\Omega) \to W^{1, p}(\mathbb{R}^n) \) and Example A.4. above to get:

**A.5. Corollary.** If \( \Omega \subset \mathbb{R}^n \) is a domain with \( C^1 \)-boundary, and if \( 1 < p < \infty \), then

\[ H := \{ u \in W^{1, p}(\Omega) : \| u \|_{L^1(\mathbb{R}^n)} \leq 1 \} \]

is relatively compact in \( C(\Omega) \).

In order to systematize the above ideas, let us recall that:

If \( X, Y \) Banach spaces and \( T: X \to Y \) a linear operator,

- \( T \) continuous / bounded if \( \| T(x) \|_Y \leq C \| x \|_X \quad \forall x \in X \)
- \( T \) compact if in addition \( T(B_X) \subset Y \) precompact (i.e. \( T(B_X) \) compact in \( Y \))
Note: Above \( B_X = \{ x \in X : ||x|| \leq 1 \} \) denotes the closed unit ball of the Banach sp. \( X \).

A.6. Definition  If \( X, Y \) Banach spaces with \( X \subseteq Y \) (and norms \( ||\cdot||_X, ||\cdot||_Y \)) say that \( X \) is compactly embedded in \( Y \), written \( X \subseteq c_c Y \), if

\[
I_{c_c}: (X, ||\cdot||_X) \to (Y, ||\cdot||_Y)
\]

is a compact operator, i.e., \( \Rightarrow \)

(a) \( ||x||_Y \leq C ||x||_X \) \( \forall x \in X \) and

(b) Each bounded sequence of \( X \) has a subsequence that converges in \( Y \), i.e., in the norm \( ||\cdot||_Y \).

Note: \( (b) \Rightarrow B_X \) rel. compact in \( Y \); indeed \( A \subseteq X \) bounded \( \Rightarrow \)

\[
A = \overline{B_X(0, M)} = M B_X(0, 1)
\]

for some \( M < \infty \); thus

\[
A = I_{c_c}(A) \subseteq M I_{c_c}(B_X) \] rel. compact when \( X \subseteq c_c Y \).

A.9. Example. \( \forall \Omega \subset \mathbb{R}^n \) open domain with \( C^1 \)-boundary \( \partial \Omega \),

\[
W^{1, p}(\Omega) \subseteq C(\overline{\Omega})
\]

by Corollary A.5 / p. 6 e

and Remark 10/ p. 5c.
A Remark on compact embeddings of $W^{1,p}(\Omega)$

Then suppose $\Omega \subset \mathbb{R}^n$ odd domain with $C^1$ boundary $\partial \Omega$.

Then

a) $W^{1,p}(\Omega) \subset C^0(\bar{\Omega})$ if $1 \leq p < n$ and $q < p^*$.

b) $W^{1,p}(\Omega) \subset L^q(\Omega)$ if $1 \leq q < \infty$ if $n \leq p < \infty$.

Proof: a) is Rellich - Kondrachov Thm; Evans Thm 7.5.7.

b) Since $\Omega$ bounded, $W^{1,p}(\Omega) \subset W^{1,s}(\Omega)$ if $s < p$.

Hence if $s < n \leq p$ and

$1 \leq q < \infty$ is given, note that $s^* = \frac{n}{n-s} \to \infty$

when $s \to n$; thus can choose $s < n$ with $q < s^*$.

In particular,

$W^{1,p}(\Omega) \subset W^{1,s}(\Omega) \subset C^0(\bar{\Omega})$.

\[ \text{Rellich - Kondrachov} \]

\[ \text{compact embedding} \]
On weak convergence and weak compactness.

1. If \( E \) is a Banach space, let \( B_E = \{ x \in E : \| x \| \leq 1 \} \) be the closed unit ball of \( E \).

2. The dual \( E^* = \{ x^* : E \rightarrow \mathbb{C}, x^* \text{ linear and continuous}, \| x^* \| = \sup \{ |<x^*, x>| : \| x \| \leq 1 \} \} \).

For our purposes, can think \( E = L^p(\Omega), 1 \leq p < \infty, \) so that (can isometrically identify) \( E^* = L^q(\Omega), \frac{1}{p} + \frac{1}{q} = 1 \).

In this case,

\[ <g^*, f> = \int g^*(x) f(x) \, dx \quad \text{for } f \in L^p(\Omega), \]
\[ g^* \in [L^p(\Omega)]^* = L^q(\Omega). \]

3. A sequence \((x_n)\) in \( E \) converges weakly to \( x \in E \), write \( x_n \rightharpoonup x \), if

\[ \lim_{n \to \infty} <x^*, x_n> = <x^*, x> \quad \forall x^* \in E^*. \]

In case of \( L^p(\Omega) \), \( f_n \rightharpoonup f \), if

\[ \int g(x) f_n(x) \, dx \to \int g(x) f(x) \, dx \quad (n \to \infty) \quad \forall g \in L^q(\Omega). \]
Weak topology = topology induced by the family $\{x^* \in E^* \}$, see [Vându, Topologia II].

Then

$$x_n \xrightarrow{w} x \iff x_n \text{ converges in the weak topology to } x.$$ 

Note: If $E^*$ separable $\exists \{x^*_n : n \in \mathbb{N}\} = E^*$ dense, i.e. $\overline{\{x^*_n \}} = E^*$

then on $B$ the weak topology is metrizable.

Metrize:

$$d(x, y) = \sum_{n=1}^{\infty} 2^{-n} \frac{|\langle x^*_n, x - y \rangle|}{1 + |\langle x^*_n, x - y \rangle|}$$

In particular, this holds in $L^p(\mathbb{R})$.

Reflective Banach spaces:

- $\forall x \in E$, $x^* \mapsto \langle x^*, x \rangle$ continuous & linear.

That is, $E \subset (E^*)^* = E^{**}$ (dual of $E$)

- $E$ reflective $\iff$ def.

Example:

If $1 < p < \infty$, then $L^p(\mathbb{R})^* = L^q(\mathbb{R})$, $\frac{1}{p} + \frac{1}{q} = 1$

$\Rightarrow$

$L^p(\mathbb{R}) = L^q(\mathbb{R})^* = L^p(\mathbb{R})^{**}$ is reflexive!

[But $L^1(\mathbb{R}), L^\infty(\mathbb{R})$ are not reflexive]
Alaoglu's theorem

If $E$ is reflexive, then $B_E$ is compact in the weak topology.

Remark: The general version of Alaoglu's theorem considers dual Banach spaces with so called $w^*$-topology.

Corollary: If $E$ is separable and reflexive, then $B_E$ is

- compact in the weak topology.
- metrizable

- sequentially compact

Thus: If $\{x_m\} \subseteq E$ bounded, can find $x \in E$ s.t.

\[ x_m \rightharpoonup x \text{ for a subsequence } \exists x_{n_k} \]

Important Consequence: If $1 < p < \infty$ and $f \in L^p(S)$, let $\{f_n\} \subseteq L^p(S)$ be any sequence with $\|f_n\|_{L^p(S)} \leq C_0 < \infty \forall n$. Then can find $g \in L^q(S)$ and a subsequence $\{f_{n_k}\}$ so that $f_{n_k} \rightharpoonup f$, i.e.

\[
\frac{1}{S} \int f_{n_k}(x)g(x) \, dx \to \frac{1}{S} \int f(x)g(x) \, dx \quad \forall g \in L^q(S).
\]
A lemma on weak limits in \( L^p(S) \)

\[ \text{Lemma: Suppose } \| f_k \|_{L^p(S)} \leq M_0 < \infty, \quad k = 1, 2, 3, \ldots \]
and \( |S| < \infty \).

If \( f_k \overset{w}{\to} f \) in \( L^p(S) \), for some \( 1 \leq p < \infty \),
then \( \| f \|_{L^q(S)} \leq M_0 \).

Note: Norm bounds remain under weak limits!

Proof of lemma: If \( f \in L^q(S) \), \( \frac{1}{p} + \frac{1}{q} = 1 \), then

\[
\int_S f_k(x) g(x) \, dx \overset{(k \to \infty)}{\to} \int_S f(x) g(x) \, dx
\]

in particular,

\[
\left| \int_S f(x) g(x) \, dx \right| \leq \lim_{k \to \infty} \int_S |f_k(x)| g(x) \, dx \leq M_0 \int_S |g(x)| \, dx
\]

Let \( E_\varepsilon := \{ x \in S : \left| f(x) \right| \geq (1 + \varepsilon) M_0 \} \) \( (\varepsilon > 0) \).

If \( |E_\varepsilon| > 0 \), set \( g(x) = \frac{|f(x)|}{|f(x)|} 1_{E_\varepsilon} \). Then

\[
M_0 \int_{E_\varepsilon} |g(x)| \, dx = M_0 |E_\varepsilon|, \quad \text{but}
\]

\[
\left| \int_S f(x) g(x) \, dx \right| = \left| \int_{E_\varepsilon} f(x) g(x) \, dx \right| \geq (1 + \varepsilon) M_0 |E_\varepsilon|
\]

And this contradicts \( \times \)!! Then \( |E_\varepsilon| = 0 \) and \( \| f \|_\infty \leq M_0 \).

Question: Can you generalize above to \( \| f_k \|_{L^q(S)} \leq M_0, \quad s \geq p \)?
Finding weak solutions via Riesz representation theorem

We look for weak solutions \( u \in W_0^{1,2}(\Omega) \) to

\[
(A) \quad Lu := -\sum_{i,j=1}^{\infty} \partial_{x_i x_j} \left( a_{ij}(x) \partial_{x_j} u \right) + c(x) u = f \quad \text{in} \; \Omega
\]

i.e.

\[-\nabla \cdot a(x) \nabla u + c(x) u = f \quad \text{in} \; \Omega,
\]

where \( a(x) \) are uniformly elliptic symmetric matrices,

i.e. \( \exists 0 < \lambda \leq \Lambda < \infty \) s.t.

\[
\lambda \| \xi \|^2 \leq \sum_{i,j=1}^{\infty} a_{ij}(x) \xi_i \xi_j \leq \Lambda \| \xi \|^2 \quad \forall \xi \in \mathbb{R}^m,
\]

and \( c(x) \in L^\infty(\Omega) \) and \( f \in L^2(\Omega) \).

Remark: Requirement \( u \in W_0^{1,2}(\Omega) \) means we are looking for weak solutions to the boundary value problem

\[
\sum_{i,j=1}^{\infty} a_{ij}(x) \partial_{x_j} u + c(x) u = f \quad \text{on} \; \Gamma
\]

\[
\| u \|_{L^2(\Omega)} = 0
\]

Riesz representation theorem: If \( H \) Hilbert space with inner product \( \langle \cdot, \cdot \rangle \) and \( F: H \to \mathbb{C} \) is linear and continuous, then there is a unique \( w \in H \) such that

\[ F(h) = \langle h, w \rangle, \quad \forall h \in H. \]
We also need a form of Poincaré inequality.

**Lemma** If \( \Omega \subset \mathbb{R}^n \) hold domain and \( u \in W_{0}^{1,2}(\Omega) \), then

\[
\int_{\Omega} u^2 \, dx \leq c(n) \operatorname{diam}(\Omega)^2 \int_{\Omega} |Du|^2 \, dx
\]

**Proof:** We know from lectures & Exercises 5 that

\[
W_{0}^{1,2}(\Omega) \subset L^2(\Omega), \quad \text{i.e.} \quad \| u \|_{L^2(\Omega)} \leq c \| u \|_{W_{0}^{1,2}(\Omega)}.
\]

To estimate \( c \), note that we have some \( c_0 < \infty \), \( \forall \beta 
\]

\[
\int_{B(0,1)} u^2 \, dx \leq c_0 \int_{B(0,1)} |Du|^2 \, dx, \quad u \in W_{0}^{1,2}(B(0,1))
\]

By scaling, i.e., considering \( u(\beta x) \), we get

\[
\int_{B(0,\beta)} u^2 \, dx \leq c_0 \beta^2 \int_{B(0,\beta)} |Du|^2 \, dx, \quad u \in W_{0}^{1,2}(B(0,\beta))
\]

By translating this holds in every ball \( B(x_0, R) \), since the zero-extension of \( u \in W_{0}^{1,2}(\Omega) \) is in \( W_{0}^{1,2}(B(x_0, R)) \) when \( \Omega = B(x_0, R) \), the claim follows from choosing \( R = \operatorname{diam}(\Omega) \).

To use these results to solving (A) define a new inner product in \( W_{0}^{1,2}(\Omega) \),

\[
< u, v > = \int_{\Omega} \left( \sum_{i,j=1}^{n} \alpha_{ij}(x) D_i u D_j v + c(x) u v \right) \, dx
\]

That \( < u, v > \) is indeed an inner product follows from
D. Lemma: There is a constant $c_0 < 0$

such that if $c < c_0$, then $\langle \cdot, \cdot \rangle$ is a positive definite inner product on $W^{1,2}_0(\Omega)$.

Proof: Write $\| u \|^2 = \langle u, u \rangle$. Here

$$\langle u, u \rangle = \int \left( \sum_{i,j=1}^n a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} + c u^2 \right) \, dx$$

$$\geq \lambda \int \left( \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_i} \right) \, dx + c_0 \int u^2 \, dx$$

By ellipticity

$$\geq x \int \frac{1}{2} \left( \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_i} \right) \, dx + c_0 \int u^2 \, dx$$

Since Lemma 8

$$\geq x \| u \|^2_{W^{1,2}_0(\Omega)} + c_0 \| u \|^2_{W^{1,2}_0(\Omega)}$$

Here $x = \min \left\{ \frac{1}{2} \frac{\lambda}{2 cn \text{ dia}(\Omega)^2} + c_0 \right\} > 0$ if

$$\langle u, u \rangle = 0 \Rightarrow u = 0 \text{ by the other properties of an inner product are clear.} \quad \Box$$

Remark: As $\| u \|^2 = \langle u, u \rangle = \int \left( \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_i} + 2 c \| u \|_{L^\infty(\Omega)}^2 \right) \, dx$

$$\leq \max \left\{ \frac{x}{\lambda}, c \| u \|_{L^\infty(\Omega)}^2 \right\} \| u \|^2_{W^{1,2}_0(\Omega)}$, the norms $\| u \|$ and $\| u \|_{W^{1,2}_0(\Omega)}$ are equivalent!
**Theorem**: There exists a constant \( c_0 < 0 \) such that the PDE

\[
\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j} \left( a_{ij}(x) \frac{\partial u}{\partial x_i} \right) + c(x) u = f
\]

has a unique weak solution \( u \in W^{1,2}_0(\Omega) \) for every \( f \in L^2(\Omega) \) provided \( c(x) \geq c_0 \).

**Remark**: \( c_0 = -\frac{c(x_0)}{\text{det}(a(x))} \) for some \( c(x_0) > 0 \).

**Proof of Theorem**: Let \( W^{1,2}_0(\Omega) \) be the completion of \( C^\infty_0(\Omega) \) with respect to the inner product \( \langle \cdot, \cdot \rangle \), equipped with the norm equivalent inner product \( \langle \cdot, \cdot \rangle \).

Then given \( f \in L^2(\Omega) \),

\[
F(v) := \int_\Omega \langle v(x), f(x) \rangle \, dx
\]

is a continuous linear map \( F : \overset{\sim}{W^{1,2}_0(\Omega)} \rightarrow \mathbb{R} \),

\[
|F(v)| \leq \left( \int_\Omega v^2 \, dx \right)^{1/2} \left( \int_\Omega f^2 \, dx \right)^{1/2} \leq \| v \|_{L^2(\Omega)} \| f \|_{L^2(\Omega)} \leq c \| f \|_{L^2(\Omega)} \| v \|_{W^{1,2}_0(\Omega)}.
\]

Hence, by Riesz representation, there is a unique \( \tilde{u} \in \overset{\sim}{W^{1,2}_0(\Omega)} \) such that

\[
\langle \tilde{u}, v \rangle = F(v), \quad v \in \overset{\sim}{W^{1,2}_0(\Omega)};
\]

i.e.

\[
\int_\Omega \left( \nabla \tilde{u} \cdot \nabla v + c(x) \tilde{u} v(x) \right) \, dx = \int_\Omega \langle f(x), v(x) \rangle \, dx.
\]

In particular, this holds \( \forall \, h \in C^\infty_0(\Omega) \) ! Thus \( \tilde{u} \) is a weak solution to \( \square \), and uniqueness follows from Riesz representation.
An Approximation Principle

As an introductory example to calculus of variations consider first the linear PDE

\[- \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left( a_{ij}(x) \frac{\partial u}{\partial x_j} \right) + c(x) u = f\]

where \(a_{ij}(x)\) are uniformly elliptic and \(c \in L^\infty(\Omega)\) as before.

This PDE is associated to the variational integral

\[I(v) = \frac{1}{2} \int_{\Omega} \left( \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} + c(x) u^2 \right) \, dx - \int_{\Omega} f(x) v \, dx\]

Example: For the Poisson eqn \(-\Delta u = f\), the associated variational integral is

\[I(v) = \frac{1}{2} \int_{\Omega} |\nabla v|^2 \, dx - \int_{\Omega} f(x) v \, dx\]

Definition: Given \(g \in W^{1,2}(\Omega)\) let

\[X = X(g) := \left\{ u \in W^{1,2}(\Omega) : u - g \in W^{1,2}_0(\Omega) \right\}\]

\[= \left\{ u \in W^{1,2}(\Omega) : u|_{\partial \Omega} = g \text{ in the Sobolev sense} \right\}\]

and call it the set of admissible functions.

Remark: If \(\Omega\) is odd with \(C^1\)-boundary \(\partial \Omega\), then

\[X = X(g) := \left\{ u \in W^{1,2}(\Omega) : \text{Trace}(u) = g \right\}\]
Definition. A function \( u \in W^{1,2}(\Omega) \) is a minimizer for the variational integral
\[ I(u), \text{ with respect to the family } \mathcal{A} \text{ if} \]
\[ I(u) \leq I(v) \quad \forall v \in \mathcal{A} \quad \text{(and } u \in \mathcal{A}). \]

Remark: If \( \mathcal{A} = \mathcal{A}(g) = \{ u \in W^{1,2}(\Omega) : u - g \in W^{1,2}_0(\Omega) \} \), then a minimizer \( u \) minimizes \( I(u) \) among all functions with the (same) boundary values \( g \).

Theorem. If \( u \in W^{1,2}(\Omega) \) is a minimizer, then \( u \) satisfies the variational integral
\[ I(u) = \frac{1}{2} \int \bigg[ \nabla u \cdot a(x) \nabla u + c(x) u^2 \bigg] \, dx - \int \varphi(x) u \, dx \]
for all \( \varphi \in \mathcal{A}(g) \).

Then \( u \) is a weak solution to the PDE
\[ \begin{cases} -\nabla \cdot a(x) \nabla u + c(x) u = \varphi \\ u \big|_{\partial \Omega} = g \end{cases} \]

Proof:
Let \( \varepsilon \in C_0^\infty(\Omega) \) and \( \varepsilon \in \mathbb{R} \). Then the variations of \( u \), \( u + \varepsilon \varphi \in g \in W^{1,2}(\Omega) = \mathcal{A}(g) \). Thus
\[ I(u) \leq I(u + \varepsilon \varphi) = I(\varepsilon), \text{ where} \]
\[ I(\varepsilon) = \frac{1}{2} \int \bigg[ \nabla (u + \varepsilon \varphi) \cdot a(x) \nabla (u + \varepsilon \varphi) + c(x) (u + \varepsilon \varphi)^2 \bigg] \, dx - \int \varphi (u + \varepsilon \varphi) \, dx. \]
As \( I'(0) \) has minimum at \( \varepsilon = 0 \implies I'(0) = 0 \) \([\text{exists}]\) \( \square \)

On the other hand, develop:

\[
I'(\varepsilon) = I(0) + \frac{1}{2} \int \nabla u \cdot \nabla u + c(x) u^2 \, dx - \int f(u) \, dx
\]

\[
+ \varepsilon \left( \frac{1}{2} \int \nabla u \cdot \nabla u + \nabla f \cdot \nabla u + \frac{\varepsilon}{2} \int 2c(x) u_\varepsilon \, dx - \int f' u_\varepsilon \, dx \right)
\]

\[
+ \varepsilon^2 \left( \frac{1}{2} \int \nabla \nabla \cdot \nabla u + \nabla f \cdot \nabla u + c(x) |u_\varepsilon|^2 \, dx \right)
\]

Thus, \( I'(\varepsilon) \) is smooth in \( \varepsilon \), with

\[
I'(0) = \int \nabla \phi \cdot \nabla u + c(x) \phi u \, dx - \int f \phi \, dx
\]

Since \( I'(0) = 0 \) \( \forall \psi \in C^2_c(\mathbb{R}) \) we see that \( u \) is a weak solution to the PDE in (B). Also \( u \big|_{\partial D} = 0 \) by assumption \( \square \)

---

**Formulas**

1. The PDE in (B) is called the Euler-Lagrange equation for the variational integral (A).

2. Does \( I'(\varepsilon) \) have a minimizer for any given boundary values \( g = \phi \)?

3. More general variational integrals \( \Rightarrow \)
   - Methods for their Euler-Lagrange equations!
   - When do minimizers exist for general variational integrals? \( \square \)
Weierstrass' example of a variational integral not admitting a minimum.

In 1870 Weierstrass gave the following example:

Let \( I(\omega) = \int_{-1}^{1} \left[ x \omega'(x)^2 \right] \, dx \)

Then there is no minimizer \( u \in W^{1,2}(-1, 1) \) (say)

within the admissible functions \( A = \{ u \in W^{1,2}(-1, 1), \, u(-1) = 0^+ \} \)

Namely let \( \omega(x) = \omega_\varepsilon(x) = 1 + \frac{\arctan(x/\varepsilon)}{\arctan(1/\varepsilon)}, \, x \in [-1, 1] \).

Then \( \omega_\varepsilon'(x) = \frac{1}{\arctan(1/\varepsilon)} \frac{\varepsilon}{x^2 + \varepsilon^2} \) with \( \omega \in A \),

so that

\[
I(\omega_\varepsilon) = \int_{-1}^{1} \frac{x^2}{\arctan(1/\varepsilon)} \frac{\varepsilon^2 \, dx}{(x^2 + \varepsilon^2)^{\frac{3}{2}}} = \varepsilon \int_{-1}^{1} \frac{\varepsilon \, dx}{x^2 + \varepsilon^2} = \frac{\varepsilon}{[\arctan(1/\varepsilon)]^2} \int_{-\sqrt{\varepsilon}}^{\sqrt{\varepsilon}} \frac{dy}{y^2 + 1} \to 0 \quad (\varepsilon \to 0^+) \]

Thus \( \inf \{ I(\omega): \omega \in A \} = 0 \)

But if \( u \in W^{1,2}(-1, 1) \) and \( I(u) = 0 \) \( \Rightarrow u' = 0 \) a.e.

So that \( u \) constant, \( u(-1) = u(+1) \); Thus \( u \in A \)!

and \( I(u) \) admits no minimizer in \( A \).

\( \square \)
Further results from Functional Analysis and weak convergence.

Recall that if \( X \) is a Banach space and \( X^* \) its dual, then
\[
\langle x^*, x \rangle \leq \| x \| \| x^* \| \quad \forall x \in X, \quad \forall x^* \in X^*.
\]

With Hahn–Banach theorem you find:
\[
\| x \| = \sup \left\{ \langle x^*, x \rangle : x^* \in X^*, \| x^* \| = 1 \right\}
\]
(see FA-notes on webpage, Sauers - 9.19, p. 173)

Another consequence of the Hahn–Banach theorem is

**Proposition I.** If \( M \) is a closed subspace of a Banach space \( X \), and \( x_0 \in X \) and
\[
d := \text{dist} (x_0, M) > 0,
\]
then there is a \( x^* \in X^* \) such that
\[
\| x^* \| = 1, \quad \langle x^*, x_0 \rangle = d.
\]
(see FA-notes, lecture 9.18, p. 172)

With these results we first show that \( I(x) := \| x \| \)
is weakly lower semicontinuous on \( X \).
Lema FA1 Let $X$ be a Banach space and suppose

\[ x_k \xrightarrow{w} x, \quad \text{i.e. sequence } \{x_k\} \text{ converges weakly to } x \in X. \]

Then

\[ \|x\| \leq \lim \inf_{k \to \infty} \|x_k\| \]

Proof: We use (1) from previous page: if $x^* \in X^*$ and $\|x^*\| = 1$

by weak convergence

\[ |\langle x^*, x \rangle| = \lim_{k \to \infty} |\langle x^*, x_k \rangle| \leq \lim \inf_{k \to \infty} \|x_k\| \|x^*\| \]

and using (1) again gives

\[ \|x\| = \sup_{\|x^*\| = 1} |\langle x^*, x \rangle| \leq \lim \inf_{k \to \infty} \|x_k\|. \]

Further results from FA:

Lemma FA2 If $M \subset X$ is a closed subspace of a Banach space $X$, if $x_k \in M$ for all $k$ and if $x_k \xrightarrow{w} x$ in $X$, then $x \in M$. [i.e. closed linear subspaces are closed under weak limits]

Proof: If $x \notin M$, then

\[ d := \text{dist}(x, M) \equiv \inf_{z \in M} \|x - z\| = \|x - z\| > 0. \]

By Proposition I/previous page, there is $x^* \in X^*$ so that

\[ \langle x^*, x \rangle = d \quad \text{and} \quad \|x^*\| = 1. \]

But then

\[ 0 = \lim_{k \to \infty} \langle x^*, x_k \rangle = \langle x^*, x \rangle = d, \quad \text{a contradiction! Thus } x \in M. \]

A third result from basic FA we needed was

\[ \]
Lemma FA 3 1. If \( x_k \xrightarrow{w} x \) in a Banach space \( X \), then 
\( x_k \xrightarrow{w} x \) is bounded, i.e., \( \sup_k \| x_k \| \leq C < \infty \).

Proof: This is a consequence of Banach–Steinhaus theorem (FA–metric, Prop. 7.41, p. 135). Namely, associate to each \( x_k \) a linear operator 
\[ T_k: X^* \rightarrow \mathbb{R}, \quad T_k(x^*) = \langle x^*, x_k \rangle. \]
Then \( \| T_k \| = \sup \{ |T_k(x^*)| : \| x^* \| = 1 \} = \sup \{ |\langle x^*, x_k \rangle| : \| x^* \| = 1 \} \]
\[ = \| x_k \| \quad \text{By (1), page 13a}. \]

Now Banach–Steinhaus theorem says that either 
\[ \sup_k \| T_k \| < \infty \] or there is \( x^* \in X^* \) for which \( \sup_k |\langle x^*, x_k \rangle| = \infty \).
Since \( \langle x^*, x_k \rangle \to \langle x^*, x \rangle \) \( \forall x^* \in X^* \), the latter cannot hold; 
thus \( \sup_k \| x_k \| = \sup_k \| T_k \| \leq C < \infty \). \( \square \)
Return to Dirichlet Principle.

In \((\textbf{11.4})\) Theorem we showed that if the variational int.

\[
I_D(u) := \frac{1}{2} \int \left[ \nabla u \cdot \alpha(x) \nabla u + c(x) u^2 \right] dx - \int f(x, u(x)) dx
\]

has a minimizer \(u\) in \(V(g) = \{ u \in W^{1,2}(\Omega) : u=g \in W_0^{1,2}(\Omega) \}\),
then \(u\) is a weak solution to

\[
\begin{cases}
-\nabla \cdot \alpha(x) \nabla u + c(x) u = f \\
u/\partial \nu = g
\end{cases}
\]

The natural assumptions here are, besides \(g \in W^{1,2}(\Omega)\), that

1. \(|g| \leq \beta \cdot \alpha(x) \leq \Lambda |g|^2\), \(c \in L^\infty\) and \(f \in L^2(\Omega)\),

and \(I_D(u)\) can be written as

\[
I_D(u) = \int_\Omega L(Du, u, x) \, dx \quad \text{where}
\]

\[
L(p, z, x) = \frac{1}{2} p \cdot \alpha(x) p + \frac{1}{2} c(x) z^2 - f(x) z
\]

From [Evans, \S 8.2/ Theorem 2, p. 448] we know that if \(L\) is convex in \(p\) and coercive, \(L(p, z, x) \geq \alpha \| p \|^2 - \beta\) for some constant \(\alpha > 0, \beta \geq 0\) then a minimizer exists.

However, assuming only \(f \in L^2\), the \(L\) in (3) is not held below, hence not coercive! \([\text{As in section 10}]\) in the end one needs
to assume $c(x) \geq c_0$ anyway, so the only real "problem" comes from the term $\|u\|_2$.

The case $f \in L^2(\Omega)$ requires thus a separate consideration which we provide here. The main lemma is the following:

(For simplicity consider the case $c(x) = 0$)

**Lemma D1.** If $x \in V(\Omega)$ is as on the previous page and $v \in V(\Omega)$, then

$$
\int_{\Omega} |\nabla v|^2 \, dx + \int_{\Omega} |\nabla \mu|^2 \, dx \leq c_1 I_D(v) + c_2
$$

**Proof:** Note that by unit, ellipticity,

$$(4) \quad \int_{\Omega} |\nabla v|^2 \, dx \leq \frac{2}{\lambda} \int_{\Omega} \Delta v \cdot a(x) \Delta v + \frac{2}{\lambda} \int_{\Omega} \phi(x) v(x) + \frac{2}{\lambda} \int_{\Omega} \phi(x) v(x) \, dx.
$$

where

$$
\int_{\Omega} \phi(x) |\nabla \mu(x)| \, dx \leq \frac{1}{\varepsilon} \int_{\Omega} |\phi|^2 \, dx + \varepsilon \int_{\Omega} |\nabla \mu|^2 \, dx.
$$

Moreover,

$$
\|\nabla g\|_{L^2(\Omega)} \leq \|\nabla - g\|_{L^2(\Omega)} + \|g\|_{L^2(\Omega)}
$$

where $\nabla - g \in W^{1,2}(\Omega)$ by assumption; thus we can use Poincaré's inequality [Evans, Theorem 3.17, page 263] to get

$$
\|\nabla - g\|_{L^2(\Omega)} \leq c \|D\nabla - Dg\|_{L^2(\Omega)} + c \|Dg\|_{L^2(\Omega)}
$$

Collecting the info we have

$$
\|\nabla\|_{L^2(\Omega)} \leq c \|D\nabla\|_{L^2(\Omega)} + c_1(\Omega)\|v\|_{H^2(\Omega)}
$$
by squaring have

\[ \int_{\mathbb{R}^2} u(x)^2 \, dx \leq C^2 \int_{\mathbb{R}^2} |\nabla u(x)|^2 \, dx + C_2(x) \]

and using this in (4) & (5) have

\[ \int_{\mathbb{R}^2} |\nabla u(x)|^2 \, dx \leq \frac{2}{\lambda} \mathcal{I}_D(u) + \frac{2}{\lambda} \varepsilon \frac{1}{4} \int_{\mathbb{R}^2} u(x) u(x) \, dx \]

\[ \leq \frac{2}{\lambda} \mathcal{I}_D(u) + \frac{2}{\lambda} \varepsilon \frac{1}{4} \int_{\mathbb{R}^2} |\nabla u(x)|^2 \, dx + \frac{2}{\lambda} C_2(x) + \frac{1}{\varepsilon} \int_{\mathbb{R}^2} \phi^2 \, dx \]

Let us choose here \( \varepsilon \) so small that \( \varepsilon \frac{1}{4} \lambda^{-1} C_2(x) < \frac{1}{2} \) \( \Rightarrow \)

\[ \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u(x)|^2 \, dx \leq \frac{2}{\lambda} \mathcal{I}_D(u) + C_3, \quad C_3 = \frac{2}{\lambda} C_2(x) + \frac{1}{\varepsilon} \int_{\mathbb{R}^2} \phi^2 \, dx \]

Finally, combining this with (6) proves the claim. \( \square \)

With the \( \alpha \)priori bound of Lemma D1, which replaces the coercivity assumptions, we can proceed quickly. But first we need

**Lower semi-continuity:**

**Lemma D2.** If \( u_k \rightharpoonup u \) in \( W^{1,2}(\mathbb{R}) \), then

\[ \mathcal{I}_D(u) \leq \liminf_{k \to \infty} \mathcal{I}_D(u_k) \]

**Proof:** Since

\[ 0 \leq (\nabla u_k - \nabla u) \cdot a(x)(\nabla u_k - \nabla u) = \nabla u_k \cdot a \nabla u_k + \Delta u \cdot a \Delta u - 2 \nabla u_k \cdot a \nabla u \]

we have...
\[ I_D(u_k) = \frac{1}{2} \int \nabla u_k \cdot a(x) \nabla u_k - \int g(x) u_k \, dx \]

\[ \geq - \frac{1}{2} \int \nabla u \cdot a(x) \nabla u + \int \nabla u_k \cdot a(x) \nabla u - \int g(x) u_k \, dx \]

The limits by weak convergence: \( |a(x)u| \in L^2 \) (ellipticity) & \( \gamma \in L^2 \) (assumption).

Thus \( \liminf_{k \to \infty} I_D(u_k) \geq - \frac{1}{2} \int \nabla u \cdot a(x) \nabla u + \int \nabla u_k \cdot a(x) \nabla u - \int g(x) u_k \, dx \)

\[ = + \frac{1}{2} \int \nabla u \cdot a(x) \nabla u \, dx - \int g(x) u \, dx = I_D(u). \]

**Theorem D3** \( \forall g \in W^{1,2}(\Omega) \) the variational int. \( I_D(u) \) has a minimum in \( A_{\Omega} \).

**Proof:** Note first that \( \inf \limits_{A_{\Omega}} I_D(u) < \infty \), since

\[ I_D(g) \leq \lambda \left( \| \nabla g \|_2^2 + \int |g(x)|^2 \right) \leq C \| Dg \|_2^2 + \| g \|_2^2 \| g \|_2 ^2 < \infty . \]

If \( u_k \) minimizing sequence, by Lemma D1 \( \{ u_k \} \) bounded in \( W^{1,2}(\Omega) \); thus it has a subsequence converging weakly,

\( u_k \rightharpoonup^w u \), and Lemma D2 \( \Rightarrow I_D(u) \leq \liminf \limits_{k \to \infty} I_D(u_k) = \inf \limits_{g \in A_{\Omega}} I_D(g) \)

It remains to show \( u \in A_{\Omega} \). But \( u_k - g \in W^{1,2}(\Omega) \) and
\( u_{w} - g \xrightarrow{w} u - g \), so that (136) / Lemma FA2

shows that \( u - g \in W^{1,2}_0(\Omega) \Rightarrow u \in A(g) \). \( \square \)

In particular, combining Theorem D.3 and (116) / Theorem solves the Dirichlet problem

**Corollary D.4.** If \( g \in W^{1,2}(\Omega) \) and \( f \in L^2(\Omega) \), there is a weak solution \( u \in W^{1,2}(\Omega) \) to

\[
\begin{cases}
-\nabla \cdot a(x) \nabla u = f & \text{in } \Omega \\
|_{\partial \Omega} = g & (\text{i.e. } u - g \in W^{1,2}_0(\Omega))
\end{cases}
\]

**Remark.** Since e.g. Poincaré inequality was used in Lemma B.1 only for \( W^{1,2}_0(\Omega) \) functions, no smoothness on \( \partial \Omega \) was required! i.e. Corollary D.4 works in all bounded domains \( \Omega \subset \mathbb{R}^n \).