Recall that a smooth $L(P, z, x)$ is a null Lagrangian, if either of the following two equivalent conditions hold,

a. Every $f \in C^\infty(\Omega; \mathbb{R}^m)$ satisfies the Euler-Lagrange equations

$$-\nabla_x \cdot D_P L(Df, f, x) + D_z L(Df, f, x) = 0 \quad \text{in} \quad \Omega, \quad k = 1, \ldots, m.$$ 

b. 

$$\int_\Omega L(Df, f, x) dx = \int_\Omega L(Dg, g, x) dx,$$

whenever $f, g \in C^\infty(\overline{\Omega}; \mathbb{R}^m)$ with $f = g$ on $\partial \Omega$.

1. Consider maps $f = (u, v, w) \in C^\infty(\mathbb{R}^3; \mathbb{R}^3)$ with differential matrix

$$Df(x) = \begin{pmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{pmatrix}$$

Show that the $2 \times 2$ minor $L(Df) := \det \begin{pmatrix} v_y & v_z \\ w_y & w_z \end{pmatrix} = v_y w_z - v_z w_y$ is a null-Lagrangian.

2. [Evans, Problem 8.7.7] Prove that $L(P) := \text{trace}(P^2) - \text{trace}(P)^2$ is a null Lagrangian. Here the trace of an $n \times n$ matrix $A = (a_{i,j})_{i,j=1}^n$ is defined by $\text{trace}(A) = \sum_{j=1}^n a_{jj}$.

3. [Evans, Problem 8.7.4] Assume $\eta : \mathbb{R}^n \to \mathbb{R}$ is $C^1$.

(i) Show that $L(P, z, x) := \eta(z) \det P$ is a null Lagrangian; here $P \in \mathbb{M}^{n \times n}$, $z \in \mathbb{R}^n$.

(ii) Deduce that if $f : \mathbb{R}^n \to \mathbb{R}^n$ is $C^1$, then

$$\int_\Omega \eta(f) \det(Df) dx$$

depends only on $f|_{\partial \Omega}$. 

Department of Mathematics and Statistics
Sobolev Spaces, Spring 2016
Exercise 9

Solutions to the exercises are to be returned by Thursday May 19 to Petri Ola, office D329.

Recall that a smooth $L(P, z, x)$ is a null Lagrangian, if either of the following two equivalent conditions hold,

a. Every $f \in C^\infty(\Omega; \mathbb{R}^m)$ satisfies the Euler-Lagrange equations

$$-\nabla_x \cdot D_P L(Df, f, x) + D_z L(Df, f, x) = 0 \quad \text{in} \quad \Omega, \quad k = 1, \ldots, m.$$ 

b. 

$$\int_\Omega L(Df, f, x) dx = \int_\Omega L(Dg, g, x) dx,$$

whenever $f, g \in C^\infty(\overline{\Omega}; \mathbb{R}^m)$ with $f = g$ on $\partial \Omega$.

1. Consider maps $f = (u, v, w) \in C^\infty(\mathbb{R}^3; \mathbb{R}^3)$ with differential matrix

$$Df(x) = \begin{pmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{pmatrix}$$

Show that the $2 \times 2$ minor $L(Df) := \det \begin{pmatrix} v_y & v_z \\ w_y & w_z \end{pmatrix} = v_y w_z - v_z w_y$ is a null-Lagrangian.

2. [Evans, Problem 8.7.7] Prove that $L(P) := \text{trace}(P^2) - \text{trace}(P)^2$ is a null Lagrangian. Here the trace of an $n \times n$ matrix $A = (a_{i,j})_{i,j=1}^n$ is defined by $\text{trace}(A) = \sum_{j=1}^n a_{jj}$.

3. [Evans, Problem 8.7.4] Assume $\eta : \mathbb{R}^n \to \mathbb{R}$ is $C^1$.

(i) Show that $L(P, z, x) := \eta(z) \det P$ is a null Lagrangian; here $P \in \mathbb{M}^{n \times n}$, $z \in \mathbb{R}^n$.

(ii) Deduce that if $f : \mathbb{R}^n \to \mathbb{R}^n$ is $C^1$, then

$$\int_\Omega \eta(f) \det(Df) dx$$

depends only on $f|_{\partial \Omega}$.
4. [Evans, Problem 8.7.5] If \( f : \mathbb{R}^n \to \mathbb{R}^n \) is as in Problem 3, fix \( x_0 \notin f(\partial \Omega) \). If \( r \) is so small that \( B(x_0, r) \cap f(\partial \Omega) = \emptyset \), choose a \( C^1 \)-map \( \eta \) so that 
\[
\int_{\mathbb{R}^n} \eta(z) dz = 1 \quad \text{and} \quad \eta(x) = 0 \quad \text{when} \ |x - x_0| \geq r.
\]
Define
\[
\deg(f, x_0) = \int_{\Omega} \eta(f) \det(Df) dx,
\]
the degree of \( f \) relative to \( x_0 \). Prove that the degree is an integer.

5. In geometric function theory one studies the distortion of a map \( f : \mathbb{R}^2 \to \mathbb{R}^2 \). Writing \( f = (u, v) \) and assuming that the Jacobian \( \det(Df(x)) > 0 \) is positive almost everywhere, the distortion is defined by
\[
K(f) := \frac{|\partial_x u|^2 + |\partial_y u|^2 + |\partial_x v|^2 + |\partial_y v|^2}{\det(Df)}
\]
Show that the functional \( L(Df) := K(f) \) is polyconvex; do this by first showing that \( F(x, y) = x^2/y \) is convex on \( (0, \infty) \times (0, \infty) \).

[Hint: You need to show that \( F(x, y) - F(a, b) \geq 2ab^{-1}(x - a) - ab^{-2}(y - b) \)]

**Note.** In higher dimensions the distortion of a map \( f : \mathbb{R}^n \to \mathbb{R}^n \) is defined by
\[
K(f) := \left[ \sum_{j,k=1}^{n} |\partial_{x_j} f^k|^2 \right]^{n/2} \det(Df)
\]
so that \( K(tf) = K(f) \) for all \( t \in \mathbb{R} \). Also in higher dimensions the distortion is polyconvex, but the algebra to prove this is a little more difficult.