1. Suppose \( f \in L^2(\Omega) \) and \( g \in W^{1,2}(\Omega) \), where \( \Omega \subset \mathbb{R}^n \) is a bounded domain with \( C^1 \)-boundary. Assume also that \( A(x) = (a_{i,j}(x)) \) is symmetric and uniformly elliptic, so that \( \lambda|\xi|^2 \leq \xi \cdot A(x)\xi \leq \Lambda|\xi|^2 \) for all \( \xi \in \mathbb{R}^n \).

We know from the lectures that the variational integral

\[
I(u) = \int_{\Omega} Du(x) \cdot A(x) Du(x) + f(x)u(x) \, dx
\]

has a minimizer in the set \( A(g) := \{ v \in W^{1,2}(\Omega) : v - g \in W^{1,2}_0(\Omega) \} \). Prove that \( I(\frac{1}{2}(u + v)) < \frac{1}{2}I(u) + \frac{1}{2}I(v) \) for \( u, v \in W^{1,2}(\Omega) \) unless \( u = v \) almost everywhere, and use this to show that the minimiser is unique.

2. [Evans, Problem 8.7.8] Explain why the methods studied in the lectures, i.e. Evans Chapter 8.2, will \textit{not} work for the integral representing the area of the graph of a function,

\[
I(w) = \int_{\Omega} (1 + |Dw|^2)^{1/2} \, dx,
\]

over \( A(g) = \{ w \in W^{1,2}(\Omega) : w - g \in W^{1,2}_0(\Omega) \} \) for any \( 1 \leq q < \infty \).

3. Given \( g \in W^{1,2}(\Omega) \) show that the Dirichlet problem

\[
\begin{aligned}
-\Delta u + u^3 &= 0, \\
u - g &\in W^{1,2}_0(\Omega)
\end{aligned}
\]

has at least one weak solution \( u \in W^{1,2}(\Omega) \), if \( \Omega \subset \mathbb{R}^4 \) is bounded with \( C^1 \)-boundary.

[Hint: Express \( u \) as a solution to the Euler-Lagrange equation of a suitable variational integral.]
4. If $a, b \in \mathbb{R}$ and $0 < t < 1$, define $w : \mathbb{R} \to \mathbb{R}$ by

$$w(s) = \begin{cases} 
    as, & \text{if } 0 \leq s < t, \\
    bs + t(a - b), & \text{if } t \leq s \leq 1, \\
    w(s - n) + nw(1), & \text{if } n < s \leq n + 1, \quad n \in \mathbb{N}.
\end{cases}$$

Given $0 \neq x_0 \in \mathbb{R}^n$ let then $u_k(x) = w(kx \cdot x_0)/k$.

If $\Omega \subset \mathbb{R}^n$ is a bounded domain, show that the sequence $u_k(x) := w(kx) \in W^{1,q}(\Omega)$, for every $1 \leq q \leq \infty$ and $k \in \mathbb{N}$. Furthermore, show that for $1 < q < \infty$ the sequence $\{u_k\}_{k \in \mathbb{N}}$ converges weakly in $W^{1,q}(\Omega)$, and determine its weak limit $u \in W^{1,q}(\Omega)$.

[Hint: Draw the graph of $w(s)$ and recall Problem 2 in Exercises 7]

5. Suppose $F : \mathbb{R}^n \to \mathbb{R}$ is not convex, so that there are $z_0, y_0 \in \mathbb{R}^n$ and $0 < t < 1$ so that $F(tz_0 + (1-t)y_0) > tF(z_0) + (1-t)F(y_0)$ and assume that $F$ is bounded. Let $\Omega = B(0,1)$ be the unit ball of $\mathbb{R}^n$.

Show that the variational integral

$$I(u) = \int_{\Omega} F(Du) \, dx$$

is not weakly lower semicontinuous in any $W^{1,q}(\Omega)$, $1 < q < \infty$.

[Hint: Consider first the case $tz_0 + (1-t)y_0 = 0$; use here Problem 4]