1. a) Derive the Euler-Lagrange equations for the variational integral

$$I(u) = \int_{\Omega} F(Du(x)) \, dx,$$

where $F : \mathbb{R}^n \to \mathbb{R}$ is a smooth function.

b) Consider the variational integral $I(u) = \int_{-1}^{1} [x u'(x)]^2 \, dx$ from the counterexample of Weierstrass [c.f. the notes/Section 12, on homepage]. Find the solutions to the Euler-Lagrange equation of this variational integral. [Hint: Non-constant solutions will have a singularity at $x = 0$.]

2. [Evans 8.6.1.b] Consider weakly converging sequences $(u_k)_{k=1}^{\infty} \subset L^p(0, 1)$, where $1 < p < \infty$; see notes/Section 8, on homepage. If $a, b \in \mathbb{R}$ and $0 < \lambda < 1$, let

$$u_k(x) = \begin{cases} a, & \text{if } j/k \leq x < (j + \lambda)/k, \\ b, & \text{if } (j + \lambda)/k \leq x < (j + 1)/k. \end{cases} \quad (j = 0, \ldots, k - 1)$$

[Draw a picture] Show that $(u_k)_{k=1}^{\infty}$ converges weakly to $u(x) \equiv \lambda a + (1 - \lambda) b$ in $L^p(0, 1)$.

3. [Evans 8.6.2] Find $L = L(p, z, x)$ so that the PDE

$$-\Delta u + D\phi \cdot Du = f \quad \text{in } \Omega$$

is the Euler-Lagrange equation corresponding to the functional $I(w) = \int_{\Omega} L(Dw, w, x) \, dx$. Here $\phi, f$ are given functions smooth in $\overline{\Omega}$.

4. [Evans 8.6.3] The elliptic regularisation of the heat equation is the PDE

$$\partial_t u - \Delta u - \varepsilon \partial^2_t u = 0 \quad \text{in } \Omega_T,$$

where $\varepsilon > 0$. This equation is used in the study of parabolic partial differential equations and provides a method to approximate solutions to the heat equation with more regularity.
where $\varepsilon > 0$, $\Omega_T = \Omega \times (0, T]$ and $\Omega \subset \mathbb{R}^n$. Show that (*) is the Euler-Lagrange equation corresponding to an energy integral

$$I_\varepsilon(w) = \int_{\Omega_T} L_\varepsilon(Dw, \partial_t w, w, x, t) \, dx \, dt.$$ 

[Here $Du = (\partial_{x_1} u, \ldots, \partial_{x_n} u)$ is the space gradient of $u$]

5. [Evans 6.6.2] A function $u \in W^{2,2}_0(\Omega) = H^2_0(\Omega)$ is a weak solution of the following boundary value problem for the biharmonic equation

$$(1) \quad \left\{ \begin{array}{ll} \Delta^2 u = f, & \text{in } \Omega, \\
 u = \frac{\partial u}{\partial n} = 0, & \text{on } \partial \Omega, \end{array} \right.$$ 

provided

$$\int_{\Omega} \Delta u \Delta v \, dx = \int_{\Omega} f v \, dx \quad \forall v \in W^{2,2}_0(\Omega).$$

Given $f \in L^2(\Omega)$, prove that there always exists a weak solution to (1).