Department of Mathematics and Statistics  
Sobolev Spaces, Spring 2016  
Exercise 6

Solutions to the exercises are to be returned by **Tuesday March 8** to Petri Ola, office D329.

1. Recall the continuous (i.e bounded) linear operators \( T : X \to Y \) between Banach spaces \( X \) and \( Y \), equipped with norm \( \| T \| = \sup\{\|Tx\| : \|x\| \leq 1\} \).

If \( T_k : X \to Y \) are compact linear operators and \( \| T - T_k \| \to 0 \), show that \( T : X \to Y \) is compact.  
[Hint: Recall the different characterisations of compactness in Banach spaces]

2. Let \( B = B(0,1) \subset \mathbb{R}^2 \). Then as discussed later, for \( f \in L^p(B) \)

\[
u(x) := (Tf)(x) = \int_B \log |x - y| f(y) dy
\]

is a solution to the Poisson equation \( \Delta u = f \) in \( B \). Show that for \( 2 < p < \infty \), \( T : L^p(B) \to W^{1,p}(B) \) is a continuous linear operator. Deduce that \( T \) is compact as an operator \( T : L^p(B) \to L^p(B) \).

[Hint: One approach is to prove \( \| Tf \|_{W^{1,p}(B)} \leq C \| f \|_{L^p(B)} \) first for \( f \in C_c^\infty(B) \). Another approach is to use difference quotients and Young’s inequality]

[Note: Claim true also for \( 1 \leq p \leq 2 \), but requires some more ”machinery”]

3. Suppose \( u \in W^{1,p}(\Omega) \), for some \( 1 < p < \infty \). If \( f : \mathbb{R} \to \mathbb{R} \) is Lipschitz-continuous with \( f(0) = 0 \), use difference quotients to show that \( f \circ u \in W^{1,p}(\Omega) \).

This is a (strong !) generalisation of Problem 4/Exercises 2. As an application, show that the positive part \( u^+ \in W^{1,p}(\Omega) \); here \( u^+(x) = u(x) \) if \( u(x) \geq 0 \) and \( u^+(x) = 0 \) otherwise.

4. Suppose \( 1 < s \leq p < \infty \) and \( |\Omega| < \infty \), so that \( L^p(\Omega) \subset L^s(\Omega) \). If \( \| f_k \|_{L^p(\Omega)} \leq 1, k = 1, 2, \ldots \) and if \( f_k \to f \) weakly in \( L^s(\Omega) \), show that

\( f \in L^p(\Omega) \) and \( \| f \|_{L^p(\Omega)} \leq 1 \).
[Hint: Recall the $L^p - L^q$ duality; c.f. proof of "Lemma on weak limits in $L^p(\Omega)$" in notes on course web-page]

5. (Evans, problem 5.10.11) Recall the difference quotients $D^h_j u(x)$ and the difference gradient $D^h u(x) = (D^h_1 u(x), D^h_2 u(x), \ldots, D^h_n u(x))$.

Prove that Theorem 3 in Evans/Section 5.8 does not hold at $p = 1$: That is, show by an example that if we have $\|D^h u\|_{L^1(\Omega')} \leq C$ for all $\Omega' \subset \subset \Omega$ and for all $|h| \leq \text{dist}(\Omega', \partial \Omega)/2$, it does not necessarily hold that $u \in W^{1,1}(\Omega)$.