1. Suppose $\Omega \subset \mathbb{R}^n$ is an arbitrary bounded subdomain. If $u \in W^{1,p}_0(\Omega)$, set
$$
\overline{u}(x) = \begin{cases} 
 u(x), & x \in \Omega \\
 0, & x \notin \Omega.
\end{cases}
$$
Show that $\overline{u} \in W^{1,p}(\mathbb{R}^n)$.

2. Let $0 \leq \eta \in C^\infty_c(\mathbb{R}^n)$, with $\text{supp}(\eta) \subset B(0,1)$ and $\int_{\mathbb{R}^n} \eta(x)dx = 1$, be a standard mollifier and set $\eta_\varepsilon(x) = \varepsilon^{-n} \eta(x/\varepsilon)$.

If $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, show that we have
$$
(\eta_\varepsilon * f)(x) \to f(x) \quad \text{as } \varepsilon \to 0
$$
at every Lebesgue point of $f$.
[Recall: $x$ Lebesgue point of $f$ if $\lim_{\varepsilon \to 0} \frac{1}{|B(x,\varepsilon)|} \int_{B(x,\varepsilon)} |f(y) - f(x)|dy = 0$.]

3. Suppose $\Omega \subset \mathbb{R}^n$ is a bounded domain with $C^1$-boundary $\partial \Omega$ and with trace operator $T : W^{1,p}(\Omega) \to L^p(\partial \Omega)$. If $\psi \in C^\infty(\overline{\Omega})$, show that
$$
\psi T(u) = T(\psi u), \quad \text{for all} \quad u \in W^{1,p}(\Omega).
$$

4. If $u \in W^{1,p}(\Omega)$, show that then $|u| \in W^{1,p}(\Omega)$.
[Hint: Apply Problem 4 in Exercises 2, with the function $f(x) = f_\varepsilon(x) = \sqrt{x^2 + \varepsilon^2} - \varepsilon$, and let $\varepsilon \to 0$.]

5. Show that there are bounded domains $\Omega \subset \mathbb{R}^2$ where the Gagliardo-Nirenberg-Sobolev inequality fails: At least for some $1 \leq p < n = 2$ and $p^* = \frac{2p}{2-p}$, there are functions $u \in W^{1,p}(\Omega) \setminus L^{p^*}(\Omega)$.

One possible class of such domains $\Omega$, called "rooms and corridors", is described on the next page.
Rooms and corridors. Let $\Omega \subset \mathbb{R}^2$ be a domain such as in the picture above,

$$
\Omega = \bigcup_{k=1}^{\infty} (D_k \cup P_k),
$$

where the ‘fat’ sets $D_k$, the rooms, and the ‘thin’ sets $P_k$, the corridors, $k = 0, 1, 2, \ldots$, are defined as follows:

Let first $d_k = 1 - 2^{-k}$, $k = 0, 1, 2, \ldots$, and define then the rooms as cubes

$$
D_k = (d_{2k}, d_{2k+1}) \times (-2^{-2k-2}, 2^{-2k-2})
$$

and the corridors as rectangles

$$
P_k = [d_{2k+1}, d_{2k+2}] \times (-\varepsilon_k 2^{-2k-2}, \varepsilon_k 2^{-2k-2}).
$$

[Hint for solving Problem 5: Choose $u$ to be constant $c_k$ in each room $D_k$, and let it grow linearly in each corridor $P_k$. Choose the constants $c_k$ so that $u \in L^p(\Omega) \setminus L^p(\Omega)$, and then the thinnesses $\varepsilon_k$ suitably to have $u \in W^{1,p}(\Omega)$]