Introduction to (Quantum) Fluctuation Relations

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Plan

- Classical Jarzynski and Crooks fluctuation relations
- Quantum Jarzynski and Crooks fluctuation relations
- Full counting statistics
- Levitov-Lesovik formula

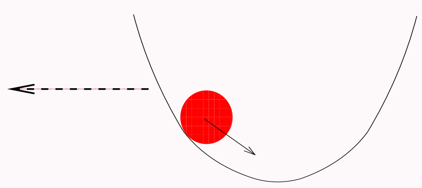
• Classical Jarzynski and Crooks fluctuation relations

- Consider the Hamiltonian evolution in the phase space $\mathcal{P} = \mathbb{R}^{2d}$ of N particles in d-dimensional space defined by the equation

$$\dot{x} \,=\, \{x, H\}$$

where $x = (q, p) = ((\vec{q}_i, \vec{p}_i)_{i=1}^N$, the Hamiltonian $H = H(t, x)$ is, in general, time-dependen, and the **Poisson** bracket is given by

$$\{F, G\} = \sum_{i=1}^{Nd} \big(\partial_{q^i} F \, \partial_{p_i} G - \partial_{p_i} F \, \partial_{q^i} G\big).$$



$$H \,=\, \frac{p^2}{2m} + \tfrac{1}{2} k (q - y(t))^2$$
$$m\ddot{q} \,=\, -k(q - y(t))$$

- Observe the evolution on the time interval $[0, \tau]$ and define the work W done on the system along trajectory $x(\cdot) \equiv X$ as given by the change of the energy of the system:

$$W[X] = H(\tau, x(\tau)) - H(0, x(0)) \equiv \Delta H[X]$$

- If we suppose that the initial condition $x(0)$ is distributed with respect to the time-zero **Gibbs** measure

$$d\mu_0(x) = \frac{1}{Z_0}\, \mathrm{e}^{-\beta H(0,x)} dx$$

then the work PDF $p(W)$ is given by

$$p(W) = \frac{1}{Z_0} \int_{\mathcal{P}} \delta(W - W[X])\, \mathrm{e}^{-\beta H(0,x(0))} dx(0)$$

- Let $x \mapsto x^*$ be the involution on $\mathcal{P}$ that changes the sign of momenta:

$$\text{if } x = (q, p) \text{ then } x^* = (q, -p)$$

Define the **time-reversed** dynamics as the one corresponding to the trajectories

$$x'(t) = x(t^*)^*$$

where $t^* = \tau - t$. We shall denote them by X'

- A straightforward check shows that

$$\dot{x}' = \{x', H'\}$$

where $H'(t, x) = H(t^*, x^*)$ so that the time-reversed dynamics is also Hamiltonian

- The work done during the time-reversed dynamics is defined the same way as for the direct one:

$$W'[X'] \;=\; H'(\tau, x'(\tau)) - H'(0, x'(0)) \;\equiv\; \Delta H'[X']$$

From the above definition it follows that

$$W'[X'] = -W[X]\,,$$

i.e. the works done during the direct and the reversed dynamics are opposite.

- If the initial point of the reversed dynamics is distributed with the **Gibbs** measure

$$d\mu_0' \;=\; \frac{1}{Z_0'}\,\mathrm{e}^{-\beta H'(0,x)}dx$$

(which is not the case if the distribution of $x'(0) = x(\tau)^*$ is induced by the direct dynamics from the **Gibbs** distribution $d\mu_0$ of its initial point) then the PDF of $W'[X']$ takes the form

$$p'(W) \;=\; \int_{\mathcal{P}} \delta(W - W'[X'])\, d\mu_0'(x'(0))$$

- We may rewrite the PDF of work in the direct dynamics in terms of the reversed dynamics:

$$\begin{aligned} p(W) &= \frac{1}{Z_0}\int_{\mathcal{P}} \delta(W - W[X])\, \mathrm{e}^{\beta\Delta H[X]}\, \mathrm{e}^{-\beta H(\tau,x(\tau))} dx(0) \\ &= \frac{1}{Z_0}\,\mathrm{e}^{\beta W}\int_{\mathcal{P}} \delta(W - W[X])\, \mathrm{e}^{-\beta H(\tau,x(\tau))} dx(\tau) \\ &= \frac{1}{Z_0}\,\mathrm{e}^{\beta W}\int_{\mathcal{P}} \delta(W + W'[X'])\; \mathrm{e}^{-\beta H'(0,x'(0))} dx'(0) \\ &= \frac{Z_0'}{Z_0}\,\mathrm{e}^{\beta W}\int_{\mathcal{P}} \delta(W + W'[X'])\, d\mu_0'(x'(0)) \\ &= \mathrm{e}^{\beta(W-\Delta F)}\, p'(-W) \end{aligned}$$

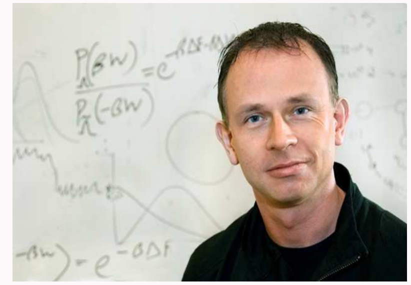
where we used the relation $W = \Delta H$, the fact that the Hamiltonian evolution and the involution $x \mapsto x^*$ preserve the **Liouville** measure dx and we introduced the free energies by the relations

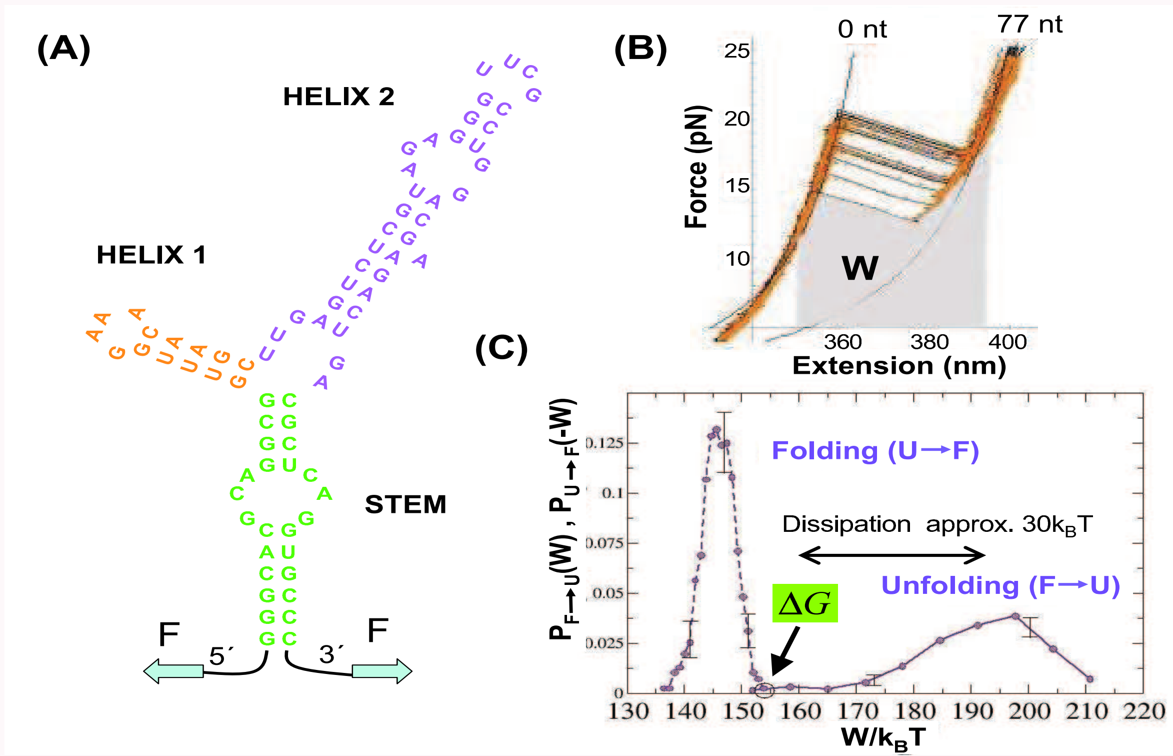
$$Z_0 = \mathrm{e}^{-\beta F_0}, \qquad Z_0' = Z_\tau = \mathrm{e}^{-\beta F_\tau}, \qquad \Delta F = F_\tau - F_0$$

- The resulting relation, that is usually written in the form

$$\frac{p(W)}{p'(-W)} = \mathrm{e}^{\beta(W-\Delta F)}$$

that assumes that $P' \neq 0$, is called the **Crooks** fluctuation relation





Statistics of work travail in stretching of an RNA molecule **Ritort**, J. Phys. C **18** (2006), R531

- The value of ΔF may be read from the point where the graphs of $p(W)$ and $p'(-W)$ intersect

- Often the time dependence of H has the form $H(\lambda(t), x)$ with $\lambda(t)$ describing the "**protocol**" of the time dependence of control parameters λ

- If $H(\lambda, x) = H(\lambda, x^*)$ (the "**micro-reversibility**") then the time reversal corresponds to the "**reversed protocol**" where the function $\lambda(t)$ is replaced by $\lambda(t^*)$

- If the protocol is symmetric under time reflection, $\lambda(t) = \lambda(t^*)$ then the direct and reversed processes coincide and in the **Crooks** relation one may replace p' by p

- In a magnetic field the micro-reversibility is understood as the relation $H(\lambda, x^*|\vec{B}) = H(\lambda, x| - \vec{B})$ and the time-reversed process involves the magnetic field with opposite direction

- **Crooks** relation implies the (historically earlier - 1997 versus 1999) **Jarzynski** equality

$$\langle \mathrm{e}^{-\beta W[X]} \rangle = \int \mathrm{e}^{-\beta W} p(W)\, dW$$
$$= \int \mathrm{e}^{\beta \Delta F} p'(-W)\, dW \;=\; \mathrm{e}^{\beta \Delta F}$$

  with the expectation $\langle \cdots \rangle$ w.r.t. the **Gibbs** distribution $d\mu_0(x(0))$ of the initial phase-space points

- **Jarzynski** equality was also used to obtain the difference ΔF of free energies from the work statistics for fast protocols rather than from the $\mathbf{2}^{\mathrm{nd}}$ **Law** inequality

$$\langle W \rangle \;\geq\; \Delta F$$

  (which it implies by the convexity of $\exp$) saturated for slow protocols where $W \simeq \Delta F$

- From the **Jarzynski** equality one obtains also the estimate

$$\int\limits_{W\leq\Delta F-w} p(W)\,dW \;\leq\; \int\limits_{W\leq\Delta F-w} \mathrm{e}^{\beta(\Delta F-w-W)}p(W)\,dW$$
$$=\; \mathrm{e}^{\beta(\Delta F-w)}\int \mathrm{e}^{-\beta W}p(W)\,dW \;\leq\; \mathrm{e}^{-\beta w}$$

  for the exponential suppression of the probability of the **2nd Law** violating realizations of the process

- One may rewrite the Crooks relation in terms of (the analytic continuation of) the characteristic functions of the PDF's $p(W)$ and $p'(W)$, i.e. their **Fourier** transforms

$$\widehat{p}(\nu) \;=\; \int \mathrm{e}^{\mathrm{i}\nu W}\,p(W)\,dW\,, \qquad \widehat{p}'(\nu) \;=\; \int \mathrm{e}^{\mathrm{i}\nu W}\,p'(W)\,dW$$

  One obtains this way the equivalent relation

$$\widehat{p}(\nu) \;=\; \widehat{p}'(\imath\beta-\nu)\;\mathrm{e}^{-\beta\Delta F}$$

• Quantum Jarzynski and Crooks fluctuation relations

- One may extract the work statistics for a closed quantum system evolving under time-dependent self-adjoint Hamiltonians $H(t)$ in the **Hilbert** space of states $\mathcal{H}$ from a two-time measurement of the energy of the system (**Kurchan** 2000)

- Assume that $H(t)$ have discrete spectrum with finite multiplicity:

$$H(t) = \sum_a e_a(t) P_a(t)$$

- Suppose that initially at time $t = 0$ the system is in the quantum **Gibbs** state represented by the density matrix

$$\rho_0 = \frac{1}{Z_0} \, \mathrm{e}^{-\beta H(0)}$$

- The first energy measurement at time $t = 0$ is performed on that state and gives the result $e_a(0)$ with probability $\mathbf{tr}(\rho_0 \, P_a(0))$ reducing the state of the system to the density matrix

$$\rho_{0+} = \frac{P_a(0) \, \rho_0 \, P_a(0)}{\mathbf{tr}(P_a(0) \, \rho_0)} = \frac{\rho_0 \, P_a(0)}{\mathbf{tr}(\rho_0 \, P_a(0))}$$

- The reduced state ρ_{0+} evolves subsequently according to the dynamics governed by the evolution operators $U(t)$ solving the **Schrödinger** equation

$$\mathrm{i}\partial_t U(t) \,=\, H(t)\, U(t)\,, \qquad U(0) = I$$

to become at time τ equal to

$$\rho_\tau \,=\, U(\tau)\, \rho_{0+} U(\tau)^{-1}$$

- The second energy measurement performed at time $t = \tau$ on this state gives the result $e_b(\tau)$ with probability $\mathbf{tr}(\rho_\tau\, P_b(\tau))$

- $\mathbf{tr}(\rho_\tau\, P_b(\tau))$ is the conditional probability of getting $e_b(0)$ in the second measurement provided that the first measurement gave result $e_a(0)$.

- The joint probability of the results $(e_a(0), e_b(\tau))$ is given by the **Bayes**' rule:

$$\begin{aligned} p_{a,b} &= \mathbf{tr}(\rho_0\, P_a(0))\, \mathbf{tr}(\rho_\tau\, P_b(\tau)) \\ &= \mathbf{tr}\big(U(\tau)\, \rho_0\, P_a(0)\, U(\tau)^{-1} P_b(\tau)\big) \\ &= \mathbf{tr}\big(\rho_0\, P_a(0)\, U(\tau)^{-1} P_b(\tau)\, U(\tau)\big) \end{aligned}$$

- This may be also rewritten as

$$p_{a,b} = \frac{1}{Z_0}\, \mathrm{e}^{-\beta\, e_a(0)}\, \mathbf{tr}\big(P_a(0)\, U(\tau)^{-1} P_b(\tau)\, U(\tau)\big)$$

- Identifying the work with the difference $e_b(\tau) - e_a(0)$ of measured energies we obtain for the work PDF the expression

$$\begin{aligned} p(W) &= \sum_{a,b} \delta(W - e_b(\tau) + e_a(0))\; p_{a,b} \\ &= \sum_{a,b} \delta(W - e_b(\tau) + e_a(0))\; \mathbf{tr}\big(\rho_0\, P_a(0)\, U(\tau)^{-1} P_b(\tau)\, U(\tau)\big) \end{aligned}$$

- For the characteristic function of $p(W)$ this gives:

$$\widehat{p}(\nu) = \int \mathrm{e}^{\mathrm{i}\nu W} p(W)\, dW = \sum_{a,b} \mathrm{e}^{\mathrm{i}\nu(e_b(\tau)-e_a(0))}\, p_{a,b}$$
$$= \sum_{a,b} \mathrm{e}^{\mathrm{i}\nu(e_b(\tau)-e_a(0))}\, \mathbf{tr}\big(\rho_0\, P_a(0)\, U(\tau)^{-1} P_b(\tau)\, U(\tau)\big)$$

which may be resummed as

$$\widehat{p}(\nu) = \mathbf{tr}\Big(\rho_0\, \mathrm{e}^{-i\nu H(0)} U(\tau)^{-1} \mathrm{e}^{i\nu H(\tau)} U(\tau)\Big)$$

- In the formulae for $p(W)$ and $\widehat{p}(\nu)$ the operator $U(\tau)$ may be replaced by the scattering one $S(\tau) = U(\tau)\, \mathrm{e}^{\mathrm{i}\tau H(0)}$ because $\mathrm{e}^{\mathrm{i}\tau H(0)}$ commutes with $\rho_0\, \mathrm{e}^{-i\nu H(0)}$

- The time reversal in quantum mechanics is realized by an anti-unitary operator θ s.t. $\theta^2 = \pm I$ with the time-reversed dynamics governed by the Hamiltonian $H'(t) = \theta\, H(t^*)\, \theta^{-1}$

- The time-reversed evolution operators $U'(t)$ solve the **Schrödinger** equation

$$\mathrm{i}\partial_t U'(t) \,=\, H'(t)\, U'(t)\,, \qquad U'(t) = 0$$

and are related to the direct evolution by the equation

$$U'(t) \,=\, \theta\, U(t^*)\, U(\tau)^{-1}\, \theta^{-1}$$

so that $U'(\tau) = \theta\, U(\tau)^{-1}\theta^{-1}$

- One has the spectral decomposition

$$H'(t) \,=\, \sum_a e'_a(t)\, P'_a(t) \,=\, \sum_a e_a(t^*)\, \theta\, P_a(t^*)\, \theta^{-1}$$

- Writing the same expressions as before for the time-reversed process starting from the **Gibbs** state

$$\rho'_0 = \frac{1}{Z'_0}\,\mathrm{e}^{-\beta H'(0)} = \frac{1}{Z_\tau}\,\theta\,\mathrm{e}^{-\beta H(\tau)}\,\theta^{-1}$$

one obtains

$$\begin{aligned}
p'_{b,a} &= \frac{1}{Z'_0}\,\mathrm{e}^{-\beta\, e'_b(0)}\ \mathbf{tr}\big(P'_b(0)\,U'(\tau)^{-1}P'_a(\tau)\,U'(\tau)\big) \\
&= \frac{1}{Z_\tau}\,\mathrm{e}^{-\beta\, e_b(\tau)}\ \overline{\mathbf{tr}\big(P_b(\tau)\,U(\tau)\,P_a(0)\,U(\tau)^{-1}\big)} \\
&= \frac{1}{Z_\tau}\,\mathrm{e}^{-\beta\, e_b(\tau)}\ \mathbf{tr}\big(P_b(\tau)\,U(\tau)\,P_a(0)\,U(\tau)^{-1}\big)^{\dagger} \\
&= \frac{1}{Z_\tau}\,\mathrm{e}^{-\beta\, e_b(\tau)}\ \mathbf{tr}\big(U(\tau)\,P_a(0)\,U(\tau)^{-1}P_b(\tau)\big) \\
&= \frac{Z_0}{Z_\tau}\,\mathrm{e}^{\beta(e_a(0)-e_b(\tau))}\,p_{a,b}
\end{aligned}$$

- Hence

$$
\begin{aligned}
p'(-W) &= \sum_{b,a} \delta(-W - (e'_a(\tau) - e'_b(0)))\, p'_{b,a} \\
&= \sum_{b,a} \delta(-W - (e_a(0) - e_b(\tau)))\, \frac{Z_0}{Z_\tau}\, \mathrm{e}^{\beta(e_a(0)-e_b(\tau))}\, p_{a,b} \\
&= \frac{Z_0}{Z_\tau}\, \mathrm{e}^{-\beta W} \sum_{a,b} \delta(W - e_b(\tau) + e_a(0))\, p_{a,b} \\
&= \mathrm{e}^{-\beta(W-\Delta F)}\, p(W)
\end{aligned}
$$

which we may rewrite as the quantum **Crooks** relation

$$
p(W) = \mathrm{e}^{\beta(W-\Delta F)}\, p'(-W)
$$

or as the relation

$$
\widehat{p}(\nu) = \widehat{p}'(\imath\beta - \nu)\, \mathrm{e}^{-\beta\Delta F}
$$

for the (analytically continued) characteristic functions

- As a corollary, one also obtains the quantum **Jarzynski** identity

$$\int \mathrm{e}^{-\beta W}\, p(W)\, dW \;=\; \mathrm{e}^{-\beta \Delta F}$$

- All these quantum relations have the same form as the classical ones and all the remarks done in the classical case apply still to the quantum situation

- For example, if micro-reversibility $\theta\, H(\lambda|\vec{B})\, \theta^{-1} = H(\lambda| - \vec{B})$ holds for a family of quantum Hamiltonians depending on control parameters λ then $H'(t) = H(\lambda(t^*)| - \vec{B})$ corresponding to the reversed protocol and the reversed magnetic field

- It should be stressed that quantum work as defined above is not a quantum observable and its reading requires two-time quantum measurements

- Non-selective measurements at intermediate times modify work statistics but preserve the quantum **Jarzynski** and **Crooks** relations (**Campisi-Talkner-Hänggi** 2010-2011)

- **Full Counting Statistics**

- The above two-time measurements may be generalized to the case where the system is composed from M subsystems initially each in an equilibrium state

$$\rho_m \,=\, \frac{1}{Z_m}\,\mathrm{e}^{-\beta^m(H^m-\sum\limits_i \mu_i^m N_i^m)}$$

corresponding to inverse temperature β^m and to chemical potentials μ_i^m of different species of particles with number operators N_i^m

- It is assumed that all the self-adjoint operators $H_m,\ N_m^i$ commute and are measured in the initial time 0 state $\rho_0=\otimes_m\rho_m$ with the results $e_a^m,\ n_{ia_i}^m$ reducing the density matrix ρ_0 to

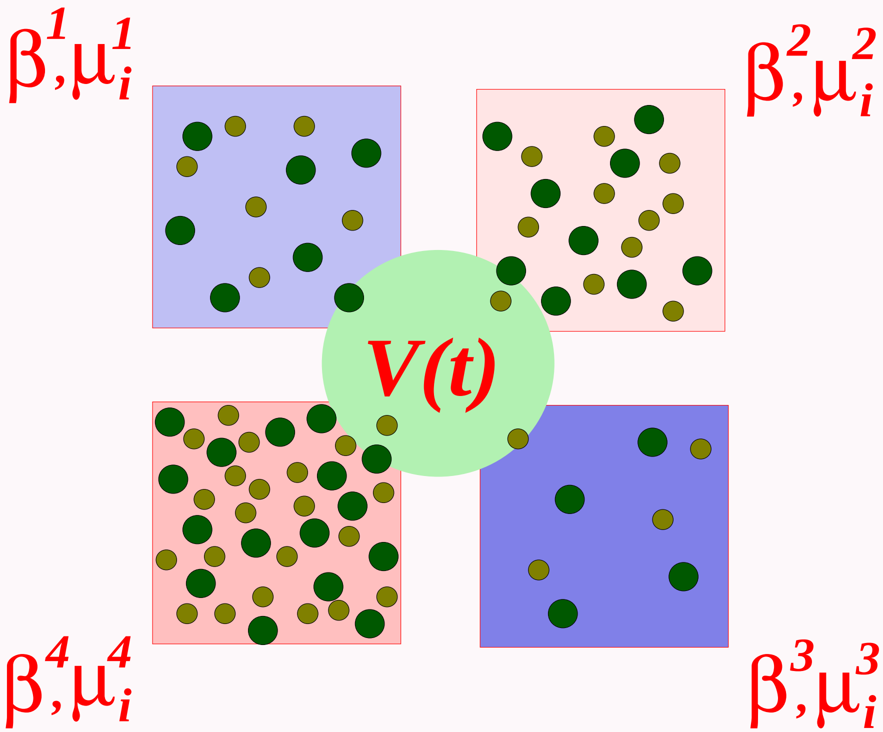
$$\rho_{0+}\,=\,P_{\vec a}\,\rho_0\,P_{\vec a}$$

where $P_{\vec a}=\bigotimes\limits_m\Big(P_a^m\prod\limits_i P_{ia_i}^m\Big)$ with $P_a^m,\ P_{i,a_i}^m$ the corresponding spectral projectors of $H^m,\ N_i^m$

- Subsequently, between time 0 and τ the system evolves with the Hamiltonian

$$H(t) = \sum_m H^m + V(t)$$

where we shall assume that $V(0) = 0 = V(\tau)$ but that it couples the subsystems at intermediate times leading to exchange of energy and particles between the subsystems



- After time τ the state of the system becomes $U(\tau)\,\rho_{0+}\,U(\tau)^{-1}$ and observables H^m, N_i^m are measured again with the results e_b^m, $n_{ib_i}^m$

- Similarly as before one obtains for the joint probability of the two-time measurements result

$$p_{\vec{a},\vec{b}} \;=\; \mathbf{tr}\Big(\rho_0\, P_{\vec{a}}\, U(\tau)^{-1}\, P_{\vec{b}}\, U(\tau)\Big)$$
$$= \prod_m \Big(\frac{1}{Z^m}\,\mathrm{e}^{-\beta^m\big(e_a^m - \sum_i \mu_i^m n_{ia_i}^m\big)}\Big)\;\mathbf{tr}\Big(P_{\vec{a}}\, U(\tau)^{-1}\, P_{\vec{b}}\, U(\tau)\Big)$$

- Denote by Δe^m, Δn_i^m the measured changes $e_b^m - e_a^m$, $n_{ib_i}^m - n_{ia_i}^m$ of energy and particle numbers and by $\boldsymbol{\Delta e}$, $\boldsymbol{\Delta n}$ their collections whose joint PDF (the "**Full Counting Statistics**") is

$$p(\boldsymbol{\Delta e},\boldsymbol{\Delta n})$$
$$= \sum_{\vec{a},\vec{b}} \prod_m \Big(\delta(\Delta e^m - e_b^m + e_a^m)\prod_i \delta(\Delta n_i^m - n_{ib_i}^m + n_{ia_i}^m)\Big)\, p_{\vec{a},\vec{b}}$$

- The characteristic function of the **FCS** takes now the form

$$\widehat{p}(\boldsymbol{\nu},\boldsymbol{\sigma}) = \int \mathrm{e}^{\,\mathrm{i}\sum\limits_m \left(\nu^m \Delta e^m + \sum\limits_i \sigma_i^m \Delta n_i^m\right)} p(\boldsymbol{\Delta e},\boldsymbol{\Delta n})\ d\boldsymbol{\Delta e}\ d\boldsymbol{\Delta n}$$
$$= \mathbf{tr}\Big(\rho_0\ \mathrm{e}^{-\mathrm{i}Q(\boldsymbol{\nu},\boldsymbol{\sigma})}\,U(\tau)^{-1}\,\mathrm{e}^{\mathrm{i}Q(\boldsymbol{\nu},\boldsymbol{\sigma})}\,U(\tau)\Big)$$

  where $Q(\boldsymbol{\nu},\boldsymbol{\sigma}) \equiv \sum\limits_m \left(\nu^m H^m + \sum\limits_i \sigma_i^m N_i^m\right)$ and again $U(\tau)$ may be replaced by $S(\tau) = U(\tau)\exp\big[\mathrm{i}\tau\sum\limits_m H^m\big]$

- We shall compare the **FCS** of energy and particle transfers for the direct process to the one for the time-reversed one

- The time reversal acts by anti-unitary operators θ^m on the subsystems and by $\theta = \bigotimes\limits_m \theta^m$ on the whole system

- To define the time-reversed system, we take:

$$(H^m)' = \theta^m H^m (\theta^m)^{-1}, \quad (N_i^m)' = \theta^m N_i^m (\theta^m)^{-1}$$

$$V'(t) = \theta V(t^*)\theta^{-1}$$

- Proceeding as before we get the fluctuation relations

$$p\big(\boldsymbol{\Delta e}, \boldsymbol{\Delta n}\big) \;=\; \mathrm{e}^{\sum\limits_m \beta^m \left(\Delta e^m - \sum\limits_i \mu_i^m \Delta n_i^m\right)}\, p'\big(-\boldsymbol{\Delta e}, -\boldsymbol{\Delta n}\big)$$

$$\widehat{p}\big(\boldsymbol{\nu}, \boldsymbol{\sigma}\big) \;=\; \widehat{p}'(-\boldsymbol{\nu} + \mathrm{i}\boldsymbol{\beta}, -\boldsymbol{\sigma} - \mathrm{i}\boldsymbol{\beta\mu})$$

  where $\boldsymbol{\beta} = (\beta^m)$, $\boldsymbol{\beta\mu} = (\beta^m \mu_i^m)$ (**Andrieux**-**Gaspard**-**Monnai** -**Tasaki** 2009)

- For a single subsystem and $\mu_i = 0$ this reduces to the **Crooks** relation for $H(0) = H(\tau)$ and $\Delta F = 0$

- **Levitov-Lesovik formula**

- In the previous setup consider subsystems composed of free non-relativistic fermions in boxes with the 1-particle space with the orthonormal basis $|\boldsymbol{k}\rangle$ with e.g. $\boldsymbol{k} \in \frac{2\pi}{L}\mathbb{Z}^d$

- Let $c^m_{\boldsymbol{k}},\ c^{m\,\dagger}_{\boldsymbol{k}}$ be the fermionic annihilation and creation operators and (taking for notational simplicity only one species of particles)

$$H^m = \sum_{\boldsymbol{k}} \epsilon^m_{\boldsymbol{k}} c^{m\,\dagger}_k c^m_{\boldsymbol{k}}\,, \qquad N^m = \sum_{\boldsymbol{k}} c^{m\,\dagger}_k c^m_{\boldsymbol{k}}$$
$$V(t) = \sum_{\boldsymbol{k},m,m'} v_{\boldsymbol{k}}(t)_{m.m'}\, c^{m\,\dagger}_{\boldsymbol{k}} c^{m'}_{\boldsymbol{k}}$$

where $v_{\boldsymbol{k}}(t)$ is a hermitian $M \times M$ matrix

- Let $u_{\boldsymbol{k}}(t)$ be the solution of the 1-particle **Schrödinger** equation

$$\mathrm{i}\partial_t u_{\boldsymbol{k}}(t) = \big(\epsilon_{\boldsymbol{k}} + v_{\boldsymbol{k}}(t)\big) u_{\boldsymbol{k}}(t)$$

and $s_{\boldsymbol{k}}(\tau) = u_{\boldsymbol{k}}(\tau)\,\mathrm{e}^{\mathrm{i}\tau\boldsymbol{\epsilon}_{\boldsymbol{k}}}$ for $\boldsymbol{\epsilon}_{\boldsymbol{k}} = \mathrm{diag}[\epsilon^m_{\boldsymbol{k}}]$

- One has

$$Q(\boldsymbol{\nu},\boldsymbol{\sigma}) = \sum_m \left(\nu^m H^m + \sigma^m N^m\right) = \sum_{\boldsymbol{k},m} \left(\nu^m \epsilon_{\boldsymbol{k}}^m + \sigma^m\right) c_{\boldsymbol{k}}^{m\,\dagger} c_{\boldsymbol{k}}^m$$

$$U(\tau)^{-1}\,\mathrm{e}^{\mathrm{i}Q(\boldsymbol{\nu},\boldsymbol{\sigma})}\,U(\tau) = S(\tau)^{-1}\,\mathrm{e}^{\mathrm{i}Q(\boldsymbol{\nu},\boldsymbol{\sigma})}\,S(\tau) = \mathrm{e}^{\mathrm{i}Q^S(\boldsymbol{\nu},\boldsymbol{\mu})}$$

where

$$Q^S(\boldsymbol{\nu},\boldsymbol{\mu}) = \sum_{\boldsymbol{k},m,m'} \left(s_{\boldsymbol{k}}(\tau)^{-1}(\boldsymbol{\nu}\boldsymbol{\epsilon}_{\boldsymbol{k}} + \boldsymbol{\sigma})s_{\boldsymbol{k}}(\tau)\right)_{m,m'} c_{\boldsymbol{k}}^{m\,\dagger} c_{\boldsymbol{k}}^{m'}$$

- Hence

$$\widehat{p}(\boldsymbol{\nu},\boldsymbol{\sigma}) = \mathbf{tr}\Big(\rho_0\,\mathrm{e}^{-\mathrm{i}Q(\boldsymbol{\nu},\boldsymbol{\sigma})}\,U(\tau)^{-1}\,\mathrm{e}^{\mathrm{i}Q(\boldsymbol{\nu},\boldsymbol{\sigma})}\,U(\tau)\Big)$$

$$= \frac{\mathbf{tr}\Big(\mathrm{e}^{\sum_{\boldsymbol{k},m} A_{\boldsymbol{k},m} c_{\boldsymbol{k}}^{m\,\dagger} c_{\boldsymbol{k}}^m}\;\mathrm{e}^{\sum_{\boldsymbol{k},m,m'} B_{\boldsymbol{k},m,m'} c_{\boldsymbol{k}}^{m\,\dagger} c_{\boldsymbol{k}}^{m'}}\Big)}{\boldsymbol{\nu} = \mathbf{0} = \boldsymbol{\sigma}}$$

where

$$A_{\boldsymbol{k},m} = -\beta^m(\epsilon_{\boldsymbol{k}}^m - \mu^m) - \mathrm{i}(\nu^m \epsilon_{\boldsymbol{k}}^m + \sigma^m)$$

$$B_{\boldsymbol{k},m,m'} = \mathrm{i}\big(s_{\boldsymbol{k}}(\tau)^{-1}(\boldsymbol{\nu}\boldsymbol{\epsilon}_{\boldsymbol{k}} + \boldsymbol{\sigma})s_{\boldsymbol{k}}(\tau)\big)_{m,m'}$$

- Since

$$\mathbf{tr}\Big(\mathrm{e}^{\sum\limits_{\boldsymbol{k},m} A_{\boldsymbol{k},m}\, c_{\boldsymbol{k}}^{m\,\dagger} c_{\boldsymbol{k}}^{m}}\ \mathrm{e}^{\sum\limits_{\boldsymbol{k},m,m'} B_{\boldsymbol{k},m,m'}\, c_{\boldsymbol{k}}^{m\,\dagger} c_{\boldsymbol{k}}^{m}}\Big) = \prod_{\boldsymbol{k}} \det\Big(I + \mathrm{e}^{\boldsymbol{A}_{\boldsymbol{k}}}\mathrm{e}^{\boldsymbol{B}_{\boldsymbol{k}}}\Big)$$

one obtains

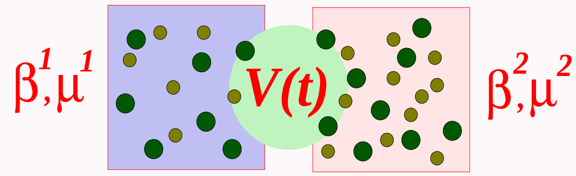
$$\widehat{p}(\boldsymbol{\nu},\boldsymbol{\sigma}) = \prod_{\boldsymbol{k}} \frac{\det\Big(I + \mathrm{e}^{-\boldsymbol{\beta}(\boldsymbol{\epsilon}_{\boldsymbol{k}}-\boldsymbol{\mu})}\mathrm{e}^{-\mathrm{i}(\boldsymbol{\nu}\boldsymbol{\epsilon}_{\boldsymbol{k}}+\boldsymbol{\sigma})}s_{\boldsymbol{k}}(\tau)^{-1}\mathrm{e}^{\mathrm{i}(\boldsymbol{\nu}\boldsymbol{\epsilon}_{\boldsymbol{k}}+\boldsymbol{\sigma})}s_{\boldsymbol{k}}(\tau)\Big)}{\det\Big(I + \mathrm{e}^{-\boldsymbol{\beta}(\boldsymbol{\epsilon}_{\boldsymbol{k}}-\boldsymbol{\mu})}\Big)} = \prod_{\boldsymbol{k}} \det\Big(I - \boldsymbol{f}_{\boldsymbol{k}} + \boldsymbol{f}_{\boldsymbol{k}}\,\mathrm{e}^{-\mathrm{i}(\boldsymbol{\nu}\boldsymbol{\epsilon}_{\boldsymbol{k}}+\boldsymbol{\sigma})}s_{\boldsymbol{k}}(\tau)^{-1}\mathrm{e}^{\mathrm{i}(\boldsymbol{\nu}\boldsymbol{\epsilon}_{\boldsymbol{k}}+\boldsymbol{\sigma})}s_{\boldsymbol{k}}(\tau)\Big)$$

where $\boldsymbol{f}_{\boldsymbol{k}} = \frac{1}{\mathrm{e}^{\boldsymbol{\beta}(\boldsymbol{\epsilon}_{\boldsymbol{k}}-\boldsymbol{\mu})}+1}$ is the diagonal matrix of **Fermi** functions

- If e.g. $\theta^m c_{\boldsymbol{k}}^m (\theta^m)^{-1} = c_{-\boldsymbol{k}}^m$ and $\epsilon_{-\boldsymbol{k}}^m = \epsilon_{\boldsymbol{k}}^m$, $v_{-\boldsymbol{k}}(t^*) = v_{\boldsymbol{k}}(t)$ then the time reversed system coincides with the original one and one has the fluctuation relation

$$\widehat{p}(\boldsymbol{\nu},\boldsymbol{\sigma}) = \widehat{p}(-\boldsymbol{\nu} + \mathrm{i}\boldsymbol{\beta}, -\boldsymbol{\sigma} - \mathrm{i}\boldsymbol{\beta}\boldsymbol{\mu})$$

- For two subsystems



$$s_{\boldsymbol{k}}(\tau) \;=\; \begin{pmatrix} r_{\boldsymbol{k}} & t'_{\boldsymbol{k}} \\ t_{\boldsymbol{k}} & r'_{\boldsymbol{k}} \end{pmatrix}$$

where $|r_{\boldsymbol{k}}|^2 + |t_{\boldsymbol{k}}|^2 = 1 = |r'_{\boldsymbol{k}}|^2 + |t'_{\boldsymbol{k}}|^2$ and $\bar{r}t' + \bar{t}r' = 0$ so that

$$\widehat{p}(\boldsymbol{\nu},\boldsymbol{\sigma}) \;=\; \prod_{\boldsymbol{k}} \Big(1 + \Big(f^1_{\boldsymbol{k}}(1-f^2_{\boldsymbol{k}})(\mathrm{e}^{\mathrm{i}(\nu^2\epsilon^2_{\boldsymbol{k}}+\sigma^2-\nu^1\epsilon^1_{\boldsymbol{k}}-\sigma^1)}-1) + f^2_{\boldsymbol{k}}(1-f^1_{\boldsymbol{k}})(\mathrm{e}^{\mathrm{i}(\nu^1\epsilon^1_{\boldsymbol{k}}+\sigma^1-\nu^2\epsilon^2_{\boldsymbol{k}}-\sigma^2)}-1)\Big)|t_{\boldsymbol{k}}|^2\Big)$$

- In the limit $\beta^1 = \beta^2 \to \infty$ one has $f^m_{\boldsymbol{k}} \to \vartheta(\mu^m - \epsilon^m_{\boldsymbol{k}})$ so that for $\epsilon^1_{\boldsymbol{k}} = \epsilon^2_{\boldsymbol{k}}$ and $\mu^1 < \mu^2$

$$\widehat{p}(\boldsymbol{\nu},\boldsymbol{\sigma}) \;=\; \prod_{\mu^1<\epsilon_{\boldsymbol{k}}<\mu^2} \Big(1 + \Big(\mathrm{e}^{\mathrm{i}\big((\nu^1-\nu^2)\epsilon_{\boldsymbol{k}}+\sigma^1-\sigma^2\big)}-1\Big)|t_{\boldsymbol{k}}|^2\Big)$$

and $\Delta e^1 + \Delta e^2 = 0 = \Delta n^1 + \Delta n^2$ with probability one

- In this case the **mean transfer** of particles and the **shot noise** are

$$\langle \Delta n^1 \rangle = \frac{\partial}{\mathrm{i}\,\partial\sigma^1} \ln \widehat{p}(\mathbf{0},\mathbf{0}) = \sum_{\mu^1 < \epsilon_{\boldsymbol{k}} < \mu^2} |t_{\boldsymbol{k}}|^2$$

$$\langle (\Delta n^1)^2 \rangle^c = \Big(\frac{\partial}{\mathrm{i}\,\partial\sigma^1}\Big)^2 \ln \widehat{p}(\mathbf{0},\mathbf{0}) = \sum_{\mu^1 < \epsilon_{\boldsymbol{k}} < \mu^2} |t_{\boldsymbol{k}}|^2 \big(1 - |t_{\boldsymbol{k}}|^2\big)$$

so that the **Fano** factor

$$\frac{\langle (\Delta n^m)^2 \rangle^c}{\langle \Delta n^m \rangle} < 1$$

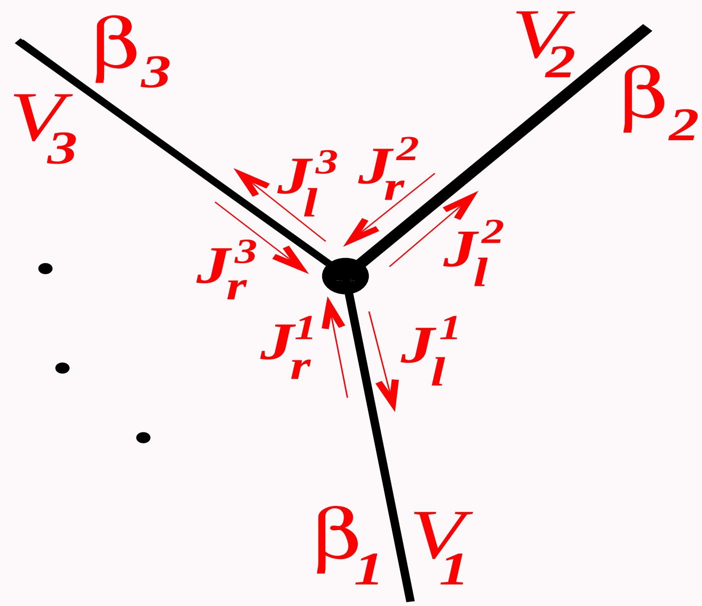
signaling the sub-Poissonian nature of the statistics of the particle transfers with the **antibunching** of exchanged particles

- If free fermions are replaced by free bosons (for which $\mu^m \geq 0$) then

$$\widehat{p}(\boldsymbol{\nu},\boldsymbol{\sigma}) = \prod_{\boldsymbol{k}} \det\Big(I + \boldsymbol{b}_{\boldsymbol{k}} - \boldsymbol{b}_{\boldsymbol{k}}\, \mathrm{e}^{-\mathrm{i}(\boldsymbol{\nu}\boldsymbol{\epsilon}_{\boldsymbol{k}}+\boldsymbol{\sigma})} s_{\boldsymbol{k}}(\tau)^{-1} \mathrm{e}^{\mathrm{i}(\boldsymbol{\nu}\boldsymbol{\epsilon}_{\boldsymbol{k}}+\boldsymbol{\sigma})} s_{\boldsymbol{k}}(\tau)\Big)^{-1}$$

where $\boldsymbol{b}_{\boldsymbol{k}} = \frac{1}{\mathrm{e}^{\boldsymbol{\beta}(\boldsymbol{\epsilon}_{\boldsymbol{k}}-\boldsymbol{\mu})}-1}$ is the matrix of **Bose**-**Einstein** functions (**Klich** 2003)

- The **FCS** has been also calculated for **Luttinger** liquids describing quantum wires at different temperatures and different potentials interacting through a junction scattering left and right currents: (**G.-Tauber** 2015)



$$J_r^m(t,0) = \sum_{m'} S^m{}_{m'} J_l^{m'}(t,0)$$

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