Bayesian inference, course exam 21.12.2017

1. Assume i.i.d sample $Y_1, \dots, Y_n$ from the negative binomial distribution Neg-bin(r_0, p), where r_0 is a known constant, with a beta prior for the parameter p:

$$\begin{aligned} Y_i \,|\, p &\sim \text{Neg-bin}(r_0, p) \quad \text{for all } i = 1, \dots, n \\ p &\sim \text{Beta}(\alpha, \beta). \end{aligned}$$

Derive the posterior distribution $p(p \,|\, \mathbf{y})$ for the parameter p given all the observations $\mathbf{y} = (y_1, \dots, y_n)$.

2. Let's consider a new observation $\tilde{Y} \sim$ Neg-bin(r_0, p) from the same distribution as the observations of the previous exercise. Assume also that this new observation is conditionally independent from the original observations given the parameter p. Denote the parameters of the posterior distribution of the previous exercise as α_n and β_n (assume the same beta prior Beta(α, β) from the previous exercise).

Show that the posterior predictive distribution for the new observation can be written as:

$$p(\tilde{y} \,|\, \mathbf{y}) = \binom{r_0 + \tilde{y} - 1}{\tilde{y}} \frac{B(r_0 + \alpha_n, \tilde{y} + \beta_n)}{B(\alpha_n, \beta_n)}.$$

3. Assume i.i.d. observations $Y_1, \dots, Y_n$ from the gamma distribution with both parameters α and β *unknown*:

$$Y_i \,|\, \alpha, \beta \sim \text{Gamma}(\alpha, \beta) \quad \text{for all } i = 1, \dots, n.$$

The conjugate prior for this model with hyperparameters $p, r, q, s > 0$ is of the form

$$p(\alpha, \beta \,|\, p, q, r, s) \propto \frac{\beta^{\alpha s}}{\Gamma(\alpha)^r} p^{\alpha - 1} \exp(-\beta q), \quad \text{where } \alpha, \beta > 0.$$

(a) Show that given this prior, the joint posterior for the parameters α and β given all the observations has the same form:

$$p(\alpha, \beta \,|\, \mathbf{y}) \propto \frac{\beta^{\alpha s_n}}{\Gamma(\alpha)^{r_n}} p_n^{\alpha - 1} \exp(-\beta q_n),$$

where the parameters of the posterior distribution are

$$\begin{aligned} p_n &:= p \prod_{i=1}^{n} y_i, \\ q_n &:= q + \sum_{i=1}^{n} y_i, \\ r_n &:= r + n, \\ s_n &:= s + n. \end{aligned}$$

(b) Show that the marginal posterior for the parameter α given all the observations $\mathbf{y} = (y_1, \dots, y_n)$ is of the form:

$$p(\alpha \,|\, \mathbf{y}) \propto \frac{\Gamma(\alpha s_n + 1)}{\Gamma(\alpha)^{r_n} q_n^{\alpha s_n + 1}} p_n^{\alpha - 1}, \quad \text{where } \alpha > 0.$$

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4. Explain briefly meanings of the following terms (you do not have to remember the exact mathematical definition for the last two items: 'hand-wavy' explanation is sufficient):

(a) Noninformative prior

(b) Probabilistic programming

(c) Strong law of large numbers

(d) Highest posterior density (HPD) credible interval

Density functions

The probability mass function of the random variable X following the **negative binomial distribution** Neg-bin(r, p) is:

$$p(x \mid r, p) = \binom{r + x - 1}{x} p^r (1 - p)^x, \quad \text{when } x = 0, 1, 2, \ldots$$

where $0 < p < 1$, and $r = 1, 2, \ldots$.

The density function of the random variable X following the **beta distribution** Beta(α, β) is:

$$p(x \mid \alpha, \beta) = \frac{1}{B(\alpha, \beta)} x^{\alpha-1}(1 - x)^{\beta-1}, \quad \text{when } 0 < x < 1,$$

where $\alpha, \beta > 0$.

The density of the random variable X following the **gamma distribution** Gamma(α, β) is:

$$p(x \mid \alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} \exp(-\beta x), \quad \text{when } x > 0,$$

where $\alpha, \beta > 0$.

Integrals

- Gaussian integral:

$$\int_{-\infty}^{\infty} e^{-x^2} \,\mathrm{d}x = \sqrt{\pi}.$$

- Gamma integral:

$$\Gamma(x) = \int_0^{\infty} x^{t-1} e^{-x} \,\mathrm{d}t.$$

- Beta function:

$$B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a + b)} = \int_0^1 t^{a-1}(1 - t)^{b-1} \,\mathrm{d}t.$$