Topology II Homework set 12

Spring 2015 Tue 21.4.2015

NB: The second course exam is planned for **Thursday, May 7th, at 13:00–15:00** with Monday, May 4th, as an alternative date. *Please send an e-mail to the lecturer as soon as possible if you wish to take the exam but cannot come at the above times.* (Let me then also know about which dates on the same week, 4.–8.5., would be possible for you.)

Exercise 1

Prove Theorem 13.17 in the textbook:

When X is a topological space and $a \in X$, let $C(a, X)$ denote the a-component of X, defined by

$$C(a, X) := \bigcup \{A \subset X \mid a \in A \text{ and } A \text{ is connected}\} \ .$$

Prove the following statements using only results preceding Section 13.16:

(a) $a \in C(a, X)$ for all $a \in X$.
(b) Every component of X is connected.
(c) The collection of the components of X forms a partition of X. (In other words, show that every $a \in X$ belongs to exactly one component of X.)
(d) If $A \subset X$ is connected and nonempty, it is contained in exactly one component of X.
(e) If $f : X \to Y$ is continuous, f maps any component of X into some component of Y.
(f) If $f : X \approx Y$, the images of the components of X are in one-to-one correspondence with the components of Y.

Exercise 2

Prove Theorem 15.3 in the textbook:

Suppose X is a topological space and endow $A \subset X$ with the relative topology inherited from X. Show that A is compact if and only if every X-open cover of A has a finite subcover.

Reminder: An X-open cover of A is a collection of open subsets $U_j \subset X$, $j \in J$, such that $A \subset \bigcup_{j \in J} U_j$.

Exercise 3

Suppose X is a locally connected and separable topological space. Show that X has only countably many components.

(Continues. . .)

Exercise 4

Prove that the projective space P^n, with $n \geq 1$, is path connected.

Reminder: The projective space was defined in Example 9.5 by identifying the "opposite points" of a sphere.

Exercise 5

Consider some $n \geq 1$ and assume that the subset $D \subset \mathbb{R}^n$ contains the origin $\mathbf{0}$ and is *open*, *bounded* and *convex*. Explain why $\mathbf{0} \notin \partial D$. Therefore, we may define a function $f : \partial D \to S^{n-1}$ by the formula $f(\mathbf{x}) := \mathbf{x}/|\mathbf{x}|$. Prove that f is a homeomorphism.

Reminder: Suppose E is a vector space. A subset $C \subset E$ is called *convex*, if to every $\mathbf{x}, \mathbf{y} \in C$ and $t \in [0, 1]$, we always have $(1 - t)\mathbf{x} + t\mathbf{y} \in C$.

(*Hint:* Starting with the plane ($n = 2$) could be helpful since then it might be easier to figure out how the three assumptions about D are used. Extending the ideas to other dimensions should then be straightforward using the results proven in the lectures and the known connectedness and compactness properties of $\mathbb{R}$ and $\mathbb{R}^n$.)