Topology II Homework set 11
Spring 2015 Tue 14.4.2015

Reminder: Due to the Easter break, the next lecture will be on Monday 13th April. There are also two weeks to work on these exercises.

Exercise 1

Prove Theorem 13.5 in the textbook:

A topological space X is said to be *disconnected* if it is the union of two disjoint nonempty open sets. Prove that the following conditions are equivalent for a topological space X:

(a) X is disconnected.
(b) $X = A \cup B$, where A and B are disjoint, nonempty and closed.
(c) There is $A \subset X$ such that $\emptyset \neq A \neq X$ and A is both open and closed.
(d) X has a *separation* $X = A|B$. (This means that $\{A, B\}$ forms a partition of X into two sets for which $\overline{A} \cap B = \emptyset = A \cap \overline{B}$.)
(e) There is a continuous surjection $f : X \to \{0, 1\}$.

(*Hint:* The corresponding metric space proof from "Topologia I" works in the general case, as well. It is possible to start by proving the results in the preceeding Theorem 13.4.)

Exercise 2

Is $\mathbb{R} \setminus \mathbb{Q}$ separable in the ordinary topology?

Exercise 3

Suppose X is a Lindelöf space and $A \subset X$ is *closed*. Prove that also A is Lindelöf in the relative topology.

Exercise 4

Suppose X and Y are topological spaces and $f : X \to Y$ is continuous.

(a) Show that if X is separable, then also fX is separable (in the relative topology inherited from Y).
(b) Show that if X is Lindelöf, then also fX is Lindelöf.

(*Remark:* The above results can be summarized by saying that "continuous functions preserve the Lindelöf property and separability.")

(Continues...)

Exercise 5

Consider an *uncountable* index set J and suppose that to every $j \in J$ there is given a Hausdorff space X_j which contains at least two points. Show that the product space $X := \prod_{j \in J} X_j$ is *not* N_1, and deduce from this that X is not metrizable nor N_2.

(*Hint:* Choose $a, b \in X$ such that $a_j \neq b_j$ for all $j \in J$. Let A denote the collection of those points $x \in X$ for which $x_j = b_j$ for finitely many j and $x_j = a_j$ for all other j. Show that $b \in \overline{A}$ and prove that no sequence of points in A can converge to b.)