Topology II Homework set 10

Spring 2015 Tue 31.3.2015

Exercise 1

Prove that the separation properties T_1 and T_2 are preserved in Cartesian products.

(In other words, given $n = 1$ or $n = 2$, you need to prove that, if X_j, $j \in J$, are T_n, then the product space $X := \prod_{j \in J} X_j$ is T_n.)

Exercise 2

Suppose that $\mathcal{B}$ is a base for a topological space X which is T_0. Prove that $\operatorname{card} X \leq \operatorname{card} \mathcal{P}(\mathcal{B})$. Conclude that if a T_0-space X has a countable base, then the cardinality of X cannot be larger than that of $\mathbb{R}$.

(*Hint:* Show that the condition "$A \in f(x) \Leftrightarrow x \in A$" defines a function $f : X \to \mathcal{P}(\mathcal{B})$. Prove that this function is injective.)

Exercise 3

Prove Theorem 12.11: The countability properties N_1 and N_2 are always inherited by subsets.

(In other words, given $n = 1$ or $n = 2$, prove that, if X is an N_n space and $A \subset X$, then A is also an N_n-space in the relative topology.)

Exercise 4

Prove that the topological space $X = \mathbb{R}$, endowed with the topology $\mathcal{T}_{\text{pa}}$ defined in Homework 2.2, is normal.

(*Hint:* Let $A, B \subset X$ be disjoint and ($\mathcal{T}_{\text{pa}}$-)closed. To every $x \in A$ find $r(x) > 0$ such that the interval $[x, x + r(x)[$ does not intersect with B. The union U of these intervals is a neighbourhood of A. Construct analogously a neighbourhood V of B and show that U and V are disjoint.)

(Continues...)

Exercise 5

Consider $X := \mathbb{R}^2$ and its "x-axis", the subset $A := \{(x,0) \mid x \in \mathbb{R}\}$. Let the collection $\mathcal{B} \subset \mathcal{P}(X)$ contain every subset which is either 1) a subset of $X \setminus A$ which is open in the ordinary topology or 2) one of the sets

$$B'(z,r) := (B(z,r) \setminus A) \cup \{z\},$$

for some $z \in A$ and $r > 0$.

(a) Using Theorem 2.9 (Base theorem) conclude that $\mathcal{B}$ is a base for a topology on X. Denote this topology by $\mathcal{T}$.
(b) Prove that $(X, \mathcal{T})$ is Hausdorff.
(c) Prove that $(X, \mathcal{T})$ is *not* regular.
(*Hint:* Consider the neighbourhood $U := B'(\mathbf{0}, 1)$ of the origin $\mathbf{0}$ and suppose V is a neighbourhood of $\mathbf{0}$ for which $\overline{V} \subset U$. Prove that this leads to a contradiction by finding a point $a = (r, 0)$, $r > 0$, for which $a \notin U$ but $a \in \overline{V}$. Recall Theorem 11.6.)