Topology II Homework set 6
Spring 2015 Tue 24.2.2015

NB: The first course exam will be held on **Thursday, March 5th, at 13:00–15:00** in one of the lecture halls in Exactum. Check the list near the entrance to Lecture hall A111 (Lars Ahlfors Auditorium) for the exact location.

The exam covers chapters 1–8 of the Topologia II textbook: more details will be given on the course webpage at least one week before the exam.

Exercise 1

Suppose that to every $j \in J$ there is given a continuous function $f_j : X_j \to X'_j$ between the topological spaces X_j and X'_j. Show that the product $g := \prod_{j\in J} f_j$, which is a function $\prod_{j\in J} X_j \to \prod_{j\in J} X'_j$ defined by the formula $g(x)_j := f_j(x_j)$, is then continuous.

Exercise 2

Prove item (2) of Theorem 7.14: Suppose $X := \prod_{j\in J} X_j$ is a product space and $A_j \subset X_j$, for $j \in J$. Show that $\overline{A} = \prod_{j\in J} \overline{A}_j$ for $A := \prod_{j\in J} A_j \subset X$.

(The point of the exercise is to prove that $\mathrm{cl}_X A$ and $\prod_{j\in J}(\mathrm{cl}_{X_j} A_j)$ are the same subset of X, *using only the results proven before Theorem 7.14*, that is, using only the basic properties of the product topology.)

Exercise 3

A function $f : X \to Y$ is called *constant*, if there is $y_0 \in Y$ such that $f(x) = y_0$ for all $x \in X$.

(a) Suppose that $Y \neq \emptyset$ is a topological space and $X \neq \emptyset$ is a set. What is the topology induced on X by the collection of all constant functions $X \to Y$?

(b) Suppose that $X \neq \emptyset$ is a topological space and $Y \neq \emptyset$ is a set. What is the topology coinduced on Y by the collection of all constant functions $X \to Y$?

(*Hint:* When is a constant function continuous?)

Exercise 4

Consider two continuous maps $f : X \to Y$ and $g : Y \to X$ between the topological spaces X and Y. Show that, if $g \circ f = \mathrm{id}$, then f is an embedding and g is an identification map.

(As in the textbook, a function is called an *identification map* (*samastuskuvaus*) if it is surjective and coinduces the original topology in the target space.)

(Continues. . .)

Exercise 5

If X is a set and $A \subset X$, the function $\chi_A : X \to \mathbb{R}$ defined by setting $\chi_A(x) = 1$, for $x \in A$, and $\chi_A(x) = 0$, for $x \notin A$, is called the *characteristic function of the set* A.

Let Y denote the product space $\mathbb{R}^{\mathbb{R}}$ and $F := \{\chi_A : \mathbb{R} \to \mathbb{R} \,|\, A \subset \mathbb{R}, A \text{ is finite}\} \subset Y$.

(a) Let g denote the constant function for which $g(x) = 1$ for all $x \in \mathbb{R}$. Show that g belongs to the closure of F.

(b) Show that no sequence in F converges to g.

(c) Explain why this implies that the topology of Y cannot be given by a metric.

(d) Construct a *discontinuous* function $f : F \cup \{g\} \to \mathbb{R}$ such that $f(x_n) \to f(x)$ whenever $x_n \to x$ in $F \cup \{g\}$.