Tomographic Reconstruction using NURBS and MCMC

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Student Seminar UH - January 30, 2015

Department of Mathematics and Statistics
Target
Target

FBP

Tikhonov
Original, TV
Original, TV and NURBS-MCMC reconstruction
Outline

- Introduction
- Background
  - NURBS
  - Tomographic Measurement Model
  - Bayesian inversion
  - MCMC
- Sugar Reconstruction
- Corrosion Pipe Reconstruction
- Conclusion
- Revisited : NURBS
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CAD (Computer-Aided Design)

NURBS (Non Uniform Rational B-Splines) the standard tool to represent geometry in CAD systems, have been the building blocks of CAD modelling.
3D-Modelling

CNC (Computer Numerical Control) system, highly automated using CAD and CAM (Computer-Aided Manufacturing).

CAD is using NURBS

Why?

Early 1970s, Pierre Bezier

Courtesy: www.aiblog.it
CAD is using NURBS

Why?

Fast in computation (small parameters)

Early 1970s, Pierre Bezier

Courtesy: www.aiblog.it
CAD is using NURBS

Why?

Fast in computation (small parameters)
Efficient!

Early 1970s, Pierre Bezier
Homogenous Object

- Take the transversal slice from the object.

- Collect the X-ray projection data.
  In other words, we have access to a collection of line integrals of the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = \begin{cases} c & \text{for } (x, y) \in \Omega, \\ 0 & \text{for } (x, y) \in \mathbb{R}^2 \setminus \Omega. \end{cases} \tag{1}$$
The angular sampling of the X-ray data is very sparse, allowing for quick measurement process (low radiation dose/ few angle data).

Our aim is to recover two things: the boundary $\partial \Omega \subset \mathbb{R}^2$ represented as a parameterized curve and the attenuation coefficient $c$. 
Implementing Bayesian Inversion and NURBS in Tomography Reconstruction

Bayesian Inversion and NURBS

→

Recovering parameters

(Control Points and attenuation value)
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NURBS curve

Video is taken from http://geometrie.foretnik.net
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Consider an attenuation function \( f : \mathbb{R}^2 \to \mathbb{R} \), \( f(x, y) \geq 0 \) and \( \text{supp}(f) \subset \Omega \) with bounded detector.

\[
\frac{dI(x)}{I(x)} = -f(x, b_1)dx,
\]

where \( I(x) \) is the intensity of the X-ray at the point \( (x, b_1) \) while passing through the source to the detector.
The radon function of the function $f$ depends on the angular parameter $\alpha$ and on a linear parameter $s \in \mathbb{R}$ as follows:

$$
\mathcal{R} f(s, \alpha) = \int_{x \cdot \vec{\alpha} = s} f(x) dx^\perp,
$$

where $dx^\perp$ is the one dimensional Lebesgue measure along the line $\{x \in \mathbb{R}^2 : x \cdot \vec{\alpha} = s\}$ and $\vec{\alpha} = \begin{bmatrix} \cos \alpha \\ \sin \alpha \end{bmatrix} \in \mathbb{R}^2$. 
Discrete Tomographic Data

In the pixel-based model, the line integral is discretized using the standard pencil-beam model. We use the pixel-based Matlab routine `radon.m` for simulating parallel-beam tomographic data.

The measurement,

\[ m_i = \int_{L_i} f(x, y) \, ds \approx \sum_{j=1}^{n} a_{ij} f_j, \]

where \( a_{ij} \) is the distance that \( L_i \) travels in the \( jth \) pixel.
NURBS-based Tomographic Model

The line integral is discretized by moving to pixel-based model using an operator $\mathcal{B} : \mathbb{R}^{2n+1} \rightarrow \mathbb{R}^{N \times N}$.

$$\mathcal{B}(v) \begin{cases} 
  c, & \text{if the pixel center is inside the NURBS curve,} \\
  0, & \text{if the pixel center is outside the NURBS curve,} 
\end{cases}$$

where $v \in \mathbb{R}^{2n+1}$. 

\[ (2) \]
Nonlinear Inverse Problem arises

Let $\mathcal{R} : \mathbb{R}^{N \times N} \to \mathbb{R}$ and $f : \mathbb{R}^2 \to \mathbb{R}$.

Consider the indirect measurement $\mathbf{m} = \mathcal{R}f + \varepsilon$, where $\mathbf{m} \in \mathbb{R}^k$ and $f = \mathcal{B}(v)$.

The inverse problem is to find $f$ which depends on $v$ when the observation, $\mathbf{m}$ is given.
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Recast inverse problem as a Bayesian inference

We use probability theory to model our lack of information in the inverse problem. All the variables in the model are considered as random variables.

- Construct a prior density (information prior to the measurement)
- Construct likelihood function (the likelihood of different between the observation and the unknown)
- Explore the posterior probability density (what we know about the unknown given observation)
Recast inverse problem as a Bayesian inference

Our model is \( m = R(B(v)) + \varepsilon \).

Let \( \varepsilon \sim N(0, \sigma^2) \), so then

\[
(m - R(B(v)) \sim N(0, \sigma^2)
\]

Model of the measurement process:

\[
\pi(m \mid v) = C \exp\left(-\frac{1}{2\sigma^2} \| R(B(v) - m \|_2^2 \right),
\]

a likelihood function.
Construct a priori information

Construct a *priori* information in a quantitative form:

Let \( v \sim \mathcal{N}(\tilde{v}, \sigma^2) \), so then

\[
\pi(v) = \exp\left(-\frac{1}{2\sigma^2} \|v - \tilde{v}\|^2_2\right),
\]

(3)

where

\[
v = \begin{bmatrix}
  r_1 \\
  \theta_1 \\
  \vdots \\
  \theta_n \\
  r_n \\
  c
\end{bmatrix}, \quad \tilde{v} = \mathbf{V} \in \mathbb{R}^{2n+1}.
\]


Construct a posterior distribution

The solution of the inverse problem is the posterior probability distribution:

\[ \pi(v | m) = \frac{\pi(v) \pi(m | v)}{\pi(m)} \]

or

\[ \pi(v | m) \sim \pi(v) \pi(m | v). \]
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Consider integral

\[ E[g(x)] = \int g(x)\pi(x)\,dx, \]

where \( \pi(x) \) is a probability density and \( g \in L^1(\mathbb{R}^n) \).
Monte Carlo Integration

Consider integral

\[ E[g(x)] = \int g(x)\pi(x)dx, \]

where \( \pi(x) \) is a probability density, and \( g \in L^1(\mathbb{R}^n) \).

In traditional Gaussian quadratures:

\[ \int g(x)\pi(x)dx \approx \sum_i^K \omega^i g(x^i), \]

a weighted sum of function values at specified points within the domain of integration, where \( \omega^i \) are the weights and \( x^i, i = 1, ..., K \) are the grid points.
Monte Carlo Integration

The Gaussian quadratures is infeasible in high dimensions. It requires $K^n$ integrations points, so then it needs a good knowledge of $\pi(x)$. 
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The idea of Monte Carlo is the grid points $x^i$ are generated randomly, choose $x^i$ to be i.i.d. samples of $\pi(x)$. 
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The idea of Monte Carlo is the grid points $x^i$ are generated randomly, choose $x^i$ to be i.i.d. samples of $\pi(x)$.

The law of large numbers:

$$\lim_{K \to \infty} \frac{1}{K} \sum_{i=1}^{K} g(x^i) = E[g(x)] = \int g(x)\pi(x)dx.$$
Monte Carlo approximates expectations with a sample average:

\[ E(p) \approx \frac{1}{n} \sum_{i=1}^{n} p_i, \]

\( p_i \) are i.i.d..

Markov chain Monte Carlo methods involve a Markov process in which a sequence of state \( p_i \) is generated. Each sample \( p_i \) has a probability distribution that depend on the previous state \( p_{i-1} \).
Metropolis Hastings

Our model is \( \mathbf{m} = \mathcal{R}(\mathcal{B}(\mathbf{v})) + \varepsilon \).

1. Set \( l = 0 \) and initialize \( \mathbf{v}(0) \).
2. Draw a random integer \( k \) from 1 to number of control points.
3. Set \( \mathbf{v} := \mathbf{v}^k + \varepsilon_k \).
4. If \( \pi(\mathbf{v}|\mathbf{m}) \geq \pi(\mathbf{v}^l|\mathbf{m}) \) then set \( \mathbf{v}^{(l+1)} := \mathbf{v} \).
5. Draw a random number \( s \) from uniform distribution on \([0, 1]\). If \( s \leq \frac{\pi(\mathbf{v}|\mathbf{m})}{\pi(\mathbf{v}^l|\mathbf{m})} \) then set \( \mathbf{v}^{(l+1)} = \mathbf{v} \), else set \( \mathbf{v}^{(l+1)} := \mathbf{v}^{(l)}. \)
6. If \( l = L \) then stop; else set \( l := l + 1 \) and go to 2nd step.
The CM (Conditional Mean) estimate is defined by

\[ \nu^{\text{CM}} = \int_{\mathbb{R}^n} \nu \pi(\nu \mid m) d\nu = E(\nu) \]

where \( \nu = \{\nu^{(l)}\}_{l=1}^L \).
The CM (Conditional Mean) estimate is defined by

\[ \nu^{CM} = \int_{\mathbb{R}^n} \nu \pi(\nu | m) d\nu = E(\nu) \]

where \( \nu = \{\nu^l\}_{l=1}^L \).

Using MCMC:

\[ \nu^{CM} \approx \frac{1}{L} \sum_{l=1}^L \nu^l. \]
Conditional Mean Estimate

The CM (Conditional Mean) estimate is defined by

$$v^{CM} = \int_{\mathbb{R}^n} v \pi(v | m) dv = E(v)$$

where $v = \{v^l\}_{l=1}^L$.

Using MCMC:

$$v^{CM} \approx \frac{1}{L} \sum_{l=1}^L v^l.$$ 

Then, we recover

$$f^{CM} = B(v^{CM}).$$
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CT Data
Setting up

Recover 12 control points $p$ and attenuation $c$ using Metropolis Hasting algorithm with 8 angles.
NURBS-MCMC reconstruction
Original and NURBS-MCMC reconstruction
Original, TV and NURBS-MCMC reconstruction
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Consider homogeneous simple corrosion pipe and set the attenuation is \( 1 \) for the pipe and \( \frac{1}{30} \) inside the pipe.
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Recovering 20 control points and the attenuation value where $N = 1000000$ and 12 angles.
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We have demonstrated that NURBS curves combining with MCMC can be used in computational inversion tomography.

- The result is automatically in CAD format (the building blocks of CAD modelling).
- The potential drawback MCMC computation is heavy (expensive) but it can be handle using parallel computing.
Conclusion

- We have demonstrated that NURBS curves combining with MCMC can be used in computational inversion tomography.
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THANK YOU FOR YOUR ATTENTION
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Revisited: NURBS
Non Uniform Rational B-Splines (NURBS)

Parametric representation of a curve and surface.

Curve

\[ S : [0, 1] \rightarrow \mathbb{R}^2. \]

They are basically piecewise polynomial functions.
Non Uniform Rational B-Splines (NURBS)

The general form of a NURBS curve is:

\[
S(t) = \frac{\sum_{i=0}^{n} P_i N_{i,p}(t) \omega_i}{\sum_{i=0}^{n} N_{i,p}(t) \omega_i} = \sum_{i=0}^{n} P_i R_{i,p}(t),
\]

where \(N_{i,p}(t)\) are B-splines basis function, \(P_i\) are the control points, \(\omega_i\) are the weights, and

\[
R_{i,p}(t) = \frac{\omega_i N_{i,p}(t)}{\sum_{i=0}^{n} \omega_i N_{i,p}(t)},
\]

are the rational B-splines basis function. The \(\omega_i \geq 0\) for all values of \(i\).
Important parts in NURBS

- **Control Point** ($P_i$)
  A set of points by which the *positions* can determine the NURBS curves.

- **Knots** ($t$)
  Defines *how much information should be shared* by segments. This vector divides the curve into intervals. The knots are needed to get the curve to settle in the proper space. A knot vector in one dimension is a set of coordinates in the parametric space, written

  $$t = \{t_1, t_2, ..., t_{n+p+1}\}.$$
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  \[ t = \{ t_1, t_2, \ldots, t_{n+p+1} \}, \]
Basis Function \( N_{i,p}(t) \)
A function which determines how strongly control point, \( P_i \) influences the curve at time \( t \).

\[
N_{i,0}(t) = \begin{cases} 
1 & \text{if } t_i \leq t < t_{i+1} \\
0 & \text{otherwise}
\end{cases}
\]

\[
N_{i,p}(t) = \frac{t - t_i}{t_{i+p} - t_i}N_{i,p-1}(t) + \frac{t_{i+p+1} - t}{t_{i+p+1} - t_{i+1}}N_{1+i,p-1}(t).
\]

Order \( (p) \)
A positive whole number plus zero, refers to the highest exponent in the polynomial basis function used for NURBS. \( p = 0, 1, 2, 3, \text{ etc.} \), refers to constant, linear, quadratic, cubic, etc., piecewise polynomials, respectively.
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Order \((p)\)

A positive whole number plus zero, refers to \textbf{the highest exponent in the polynomial basis function} used for NURBS. \(p = 0, 1, 2, 3, etc.\), refers to constant, linear, quadratic, cubic, etc., piecewise polynomials, respectively.
Example of *uniform* knot vector:

\[
[0 \ 0.25 \ 0.5 \ 0.75 \ 1.0]
\]

Some examples of *open* uniform knot vector:

\[
p = 2, \quad [0 \ 0 \ \frac{1}{4} \ \frac{1}{2} \ \frac{3}{4} \ 1 \ 1]
\]

\[
p = 3, \quad [0 \ 0 \ 0 \ \frac{1}{3} \ \frac{2}{3} \ 1 \ 1 \ 1]
\]

\[
p = 4, \quad [0 \ 0 \ 0 \ 0 \ \frac{1}{2} \ 1 \ 1 \ 1 \ 1]
\]
Knots

Formally, an *open uniform* knot vector is given by

\[
\begin{align*}
  t_i &= 0, & 0 \leq i \leq p \\
  t_i &= i - p, & p + 1 \leq i \leq n + 1 \\
  t_i &= n - p + 2, & n + 2 \leq i \leq n + p + 1
\end{align*}
\]

*Non uniform* knot vectors may have either spaced and/or multiple internal knot values. Here are the examples

\[
[0 0 0.28 0.5 0.72 1]
\]
Closed NURBS Curve

- Set the same control point in the ends by using *open uniform* knot vector.
- Repeat the \( p - 1 \) control points by using *periodic uniform* knot vector.