

# Sparse X-ray Tomography using Total Variation

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## INTRODUCTION

In this project the goal is to recover the inner structure of hazelnut with sparse X-ray imaging using total variation regularization. The arising minimization problem is solved with Barzilai-Borwein method.

The data for this project is real, sparse X-ray data, from approximately 20 angles. Since the object is very small and far from the X-ray source in measurements, the X-rays are thought to be parallel, not in fan formation.

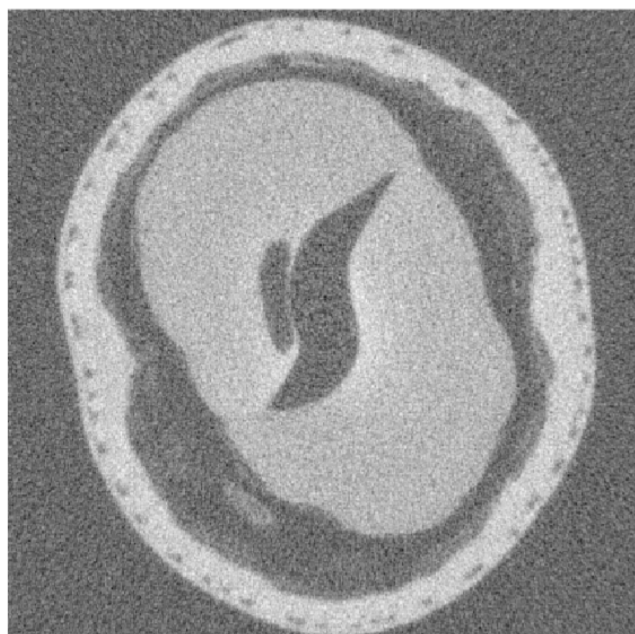


Figure 1: Image obtained by full-angle tomography.

## METHODS AND MATERIALS

When reconstructing the hazelnut, it is thought to be pixelized so that computations are possible. The reconstruction vector  $\mathbf{f}$  below is formed from the pixelized image by rearranging it to a long column.

Total variation regularization method focuses on minimizing the following formula regarding to reconstruction  $\mathbf{f}$ :

$$\|A\mathbf{f} - \mathbf{m}\|_2^2 + \alpha \sum_{j=1}^n |(L_H \mathbf{f})_j| + \alpha \sum_{j=1}^n |(L_V \mathbf{f})_j|, \quad (1)$$

where  $\mathbf{m}$  is the measured data,  $A, L_H, L_V \in \mathbb{R}^{k \times n}$ ,  $\mathbf{f} \in \mathbb{R}^n$  and  $\alpha$  is the regularization parameter.

The matrix  $A$  models how long a distance each ray travels in each pixel of the nut.  $A$  is known and easy to calculate.

The matrices  $L_V$  and  $L_H$  in 1 penalize derivative. They encourage the neighbouring pixel values (vertical and horizontal, respectively) to be as close to each other as possible with 1 and -1 values.

The reconstruction parameter  $\alpha$  is chosen using multi-resolution parameter choice method introduced in [2]. The total variation norm (TV norm) is defined

$$\frac{1}{n} \sum |\mathbf{f}_k - \mathbf{f}_{k'}|$$

where the sum is over horizontally and vertically neighbouring pixel values and the size of the pixelized picture is  $n \times n$ . The idea in the method is to choose value for  $\alpha$  so, that it is the first value where the TV norm is approximately the same for all resolutions.

With large data sets the Barzilai-Borwein minimization method is efficient. It is an iterative approach which also uses gradients. Because of that the absolute values must be replaced with differentiable approximation, namely  $|t| \rightarrow \sqrt{t^2 + \beta}$  where  $\beta$  is a small constant. The method is to calculate  $\mathbf{f}$  iteratively:

$$\mathbf{f}^{(l+1)} = \mathbf{f}^{(l)} - \delta_l \nabla G_\beta(\mathbf{f}^{(l)})$$

for

$$\delta_l = \frac{y_l^T y_l}{y_l^T g_l}$$

where  $y_l = \mathbf{f}^{(l)} - \mathbf{f}^{(l-1)}$  and  $g_l = \nabla G_\beta(\mathbf{f}^{(l)}) - \nabla G_\beta(\mathbf{f}^{(l-1)})$ , and the function we want to minimize is  $G_\beta$ .

The first steplength  $\delta_1$  is usually chosen to be  $\frac{1}{10000}$  or some other small value, and  $f^{(1)}$  can be guessed. Also the parameter  $\beta$  needs to be chosen carefully for good reconstructions.

The smaller resolution data was obtained by interpolating it from the original data.

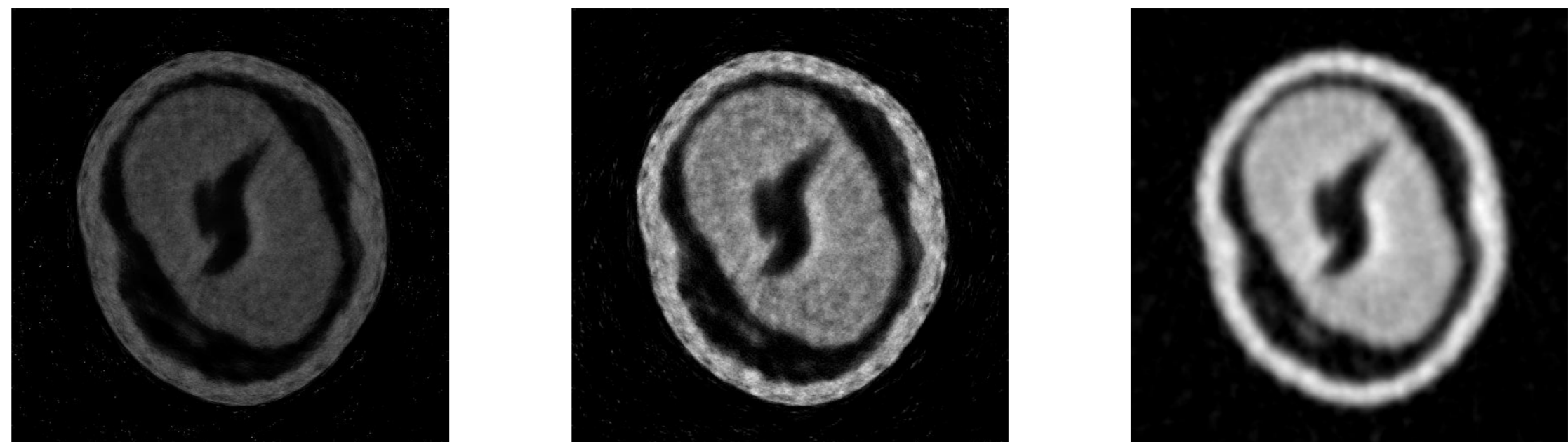


Figure 2: Reconstructions with  $\alpha = 0.0001$  (left),  $\alpha = 0.1$  (middle) and  $\alpha = 100$  (right) with resolution  $512 \times 512$ .

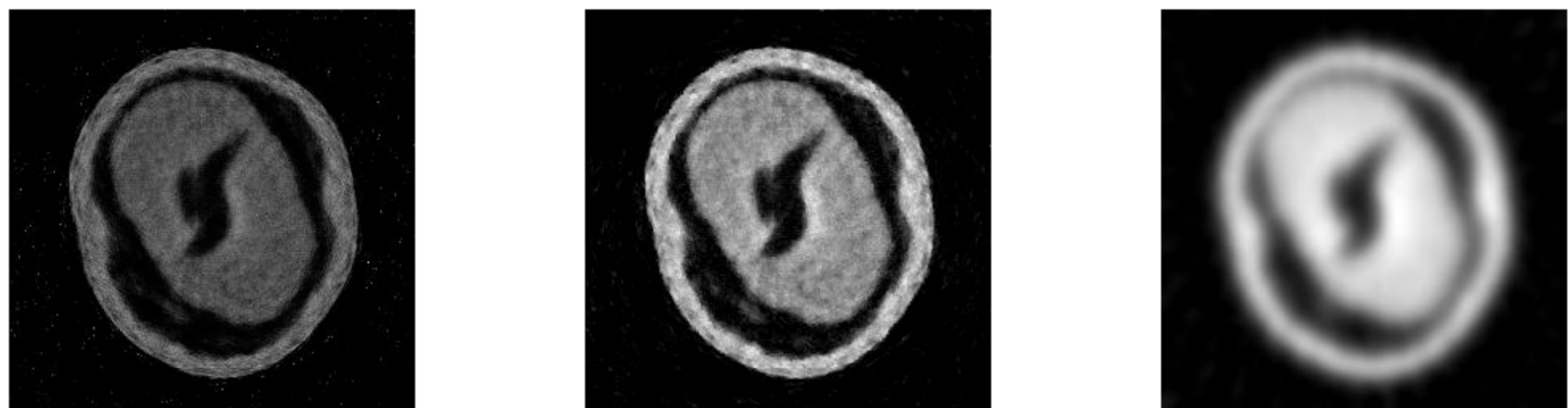


Figure 3: Reconstructions with  $\alpha = 0.0001$  (left),  $\alpha = 0.1$  (middle) and  $\alpha = 100$  (right) with resolution  $256 \times 256$ .



Figure 4: Reconstructions with  $\alpha = 0.0001$  (left),  $\alpha = 0.1$  (middle) and  $\alpha = 100$  (right) with resolution  $192 \times 192$ .

## RESULTS

The following table contains the total variation norm values from pictures of different resolutions and with different alpha values. Column  $E$  shows the mean difference between norms in a row.

$\alpha$	$512 \times 512$	$256 \times 256$	$192 \times 192$	$E$
0.0001	0.114	0.148	0.162	0.0321
0.001	0.112	0.141	0.152	0.0267
0.01	0.105	0.128	0.136	0.0205
0.1	0.089	0.096	0.097	0.0057
1	0.055	0.058	0.064	0.0060
10	0.032	0.040	0.043	0.0074
100	0.022	0.023	0.021	0.0014

The value 0.1 is chosen optimal for  $\alpha$ , since it is the first value for which the norms are approximately the same for every resolution.

In Barzilai-Borwein optimization we used constant amount of steps, namely 1000, and  $\beta = 0.000001$ .

Example reconstructions for different alpha values can be seen from figure 2.

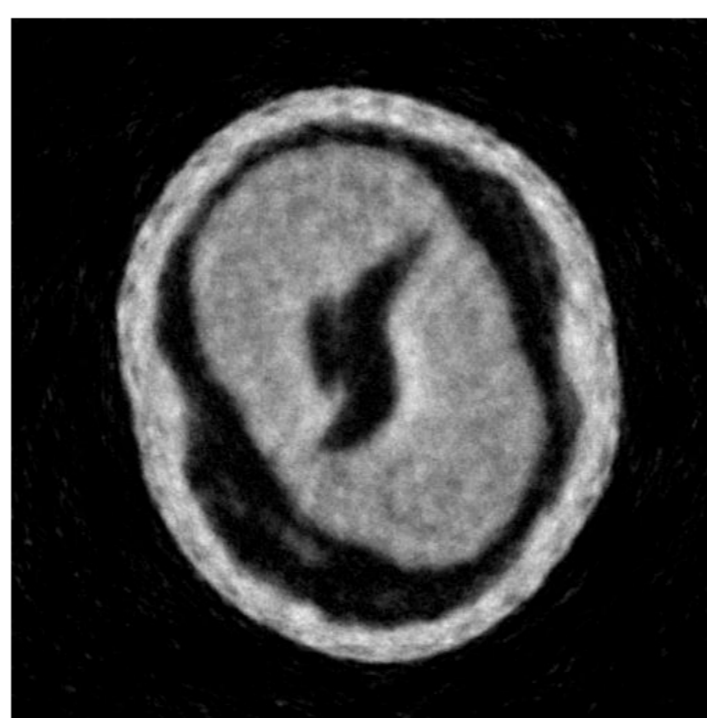


Figure 5: Reconstruction when  $\alpha = 1$  and resolution is  $512 \times 512$ .

## DISCUSSION

Image obtained from full-angle tomography for comparison can be seen from figure 1. The reconstructions with fewer angles are very similar.

The method for choosing alpha seems to work quite well. It clearly discards too large alpha values, which can be seen from the rightmost pictures of 2, 3 and 4. The pictures are very blurry. For example the small dots in the shell that can be seen from the image obtained by full-angle tomography are all gone.

On the other hand the leftmost reconstructions which had too small alpha according to our method, are grainy. It is hard to see which details are real ones and which are just from noise.

However the chosen alpha may not be optimal. In figure 5 is the reconstruction with  $\alpha = 1$ , which is quite good, perhaps even slightly better than the reconstruction with  $\alpha = 0.1$ . Better choice could probably be found by trying out more values of  $\alpha$ , but that makes the computation even more costly.

As can be seen in the table of total variation norms, the mean difference first drops with  $\alpha = 0.1$ , but then rises a bit before dropping again as  $\alpha$  gets larger. This seems a bit peculiar, but on the other hand the change is quite small and probably not too important.

## References

- [1] Mueller J.L. and Siltanen S, *Linear and Nonlinear Inverse Problems with Practical Applications*, SIAM 2012.
- [2]. Niinimäki K, Lassas M, Hämäläinen K, Kallonen A, Kolehmainen V, Niemi E and Siltanen S *Multi-resolution parameter choice method for total variation regularized tomography*, arXiv:1407.2386v1