Practical X-ray tomography by total variation reconstruction

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INTRODUCTION

The aim of this work is to practice sparse-angle 2D tomography. We consider tomographic data on a walnut taken from 20 angles with 9 degree angular steps, and our goal is to infer the attenuation of different parts of the walnut from the measurements. As a result we get a discrete ill-posed inverse problem.

We build a discretized measurement model where the attenuation of the object is modeled by a $778 \times 778$ pixel image $f$. The attenuation is assumed to be non-negative and constant in each pixel. The pixels are numbered from 1 to $N = 778^2$ and the attenuation is stored as $f \in \mathbb{R}^N$. The measurements are taken by fan beam geometry but in the calculations we assume parallel beam geometry. The number of measurement directions is $J = 20$ and the number of measurements per direction is 1105, and so the measurement data is collected in $m \in \mathbb{R}^N$, $k = 22100$.

A measurement $m_k$ gives the line integral of the attenuation $f$ over the line $L_k$, and is approximated by the sum

$$m_k = \sum_{j=1}^{N} a_{kj} f_j$$

where $a_{kj}$ is the length of the intersection of $L_k$ and the $j$th pixel. The lengths $a_{kj}$ comprise the matrix $A \in \mathbb{R}^{N \times J}$. We thus get a discrete measurement model

$$Af = m$$

where $f$ is to be determined.

Equation (1) is ill-posed, and naive least squares inversion is highly susceptible to measurement noise. The inevitable presence of noise in practical measurement forces us to use regularization instead of simply looking for a least-squares solution of problem (1) (see [MS]).

METHODS AND MATERIALS

As a reconstruction tool we use total variation (TV) regularization. In contrast to the classical filtered back-projection (FBP), total variation regularization is well-suited to sparse angle tomography. It sharpens edges in the reconstruction and simultaneously removes noise efficiently by promoting sparsity of the gradient of the attenuation. The idea is to strike a balance between minimizing the discrepancy $\|Af - m\|^2$ and minimizing the total variation of $f$.

Total variation regularization was introduced by Rudin, Osher and Fatemi in [ROF]. For more on total variation regularization in tomographic imaging see e.g. [HIHHKNS] and [MS].

In order to discretize total variation regularization recall that in MATLAB’s enumeration of pixels $f_{j_{xy}}$, is the right neighbor and $f_{j_{xy}+1}$, is the down neighbor of $f_{j_{xy}}$. We form the horizontal difference operator $L_h$ defined by $L_h f_j := f_{j+1} - f_{j-1}$ and the vertical difference operator $L_v$ given by $(L_v f_{j})_j := f_{j+1} - f_{j-1}$. The functional to be minimized is

$$G(f) = \|Af - m\|^2 + \alpha \sum_{j=1}^{N} \|L_h f_j\| + \sum_{j=1}^{N} \|L_v f_j\|,$$

where $\alpha > 0$ is a suitable regularization parameter.

The dimension of the minimization problem is so large that iterative methods are required in order to find an approximate minimizer in reasonable computation time. We use a gradient-based optimization method, and that requires us to replace $\|f\|_1$ by $\sqrt{\sum_{j=1}^{N} \|

\delta_j\|^2}$, $\beta > 0$, in the penalty term of $G$ since $\|f\|_1$ is not differentiable at zero. The objective functional to be minimized thus becomes

$$G(f) = \|Af - m\|^2 + \alpha \sum_{j=1}^{N} \|L_h f_j\| + \beta \sum_{j=1}^{N} \|L_v f_j\|.$$