Inverse problems course
Exercise 3 (February 3–5, 2015)
University of Helsinki
Department of Mathematics and Statistics
Samuli Siltanen and Andreas Hauptmann
Related book sections (Mueller & Siltanen 2012):

Theoretical exercises:

T1. Set

\[ A = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}. \]

(a) Draw the range of \( A \) as a subset of \( \mathbb{R}^2 \). Draw also the point \((0, 1)\) to the same image.

(b) Find the least-squares solution(s) of equation \( Af = m \), where \( m = [0, 1]^T \).

(c) Determine the minimum-norm solution of equation \( Af = m \) by geometric arguments. (Analyze the triangles involved.)

T2. (a) Diagonalize (by hand, not computer) the symmetric matrix \( A^T A \), where \( A \) is as above. Make sure that the eigenvectors are orthonormal.

(b) Follow the method of Problem T3 of Exercise 2 and calculate the singular value decomposition of \( A \) by hand.

(c) Find the minimum-norm solution of equation \( Af = m \) by Moore-Penrose pseudoinverse. Do you get the same answer than in Problem T1?

T3. Thin lines depict pixels and thick lines X-rays in this image:

Give a numbering to the nine pixels \((f \in \mathbb{R}^9)\) and to the six X-rays \((m \in \mathbb{R}^6)\), and construct the matrix \( A \) for the measurement model \( m = Af \). The length of the side of a pixel is one.
Matlab exercises:

M1. Consider equations $x_1 + x_2 = 1$, $x_2 = -2$ and $-\frac{1}{3}x_1 + x_2 = -2$.

(a) Write the equations in the matrix form $Ax = y$. (That is, specify the elements in the $3 \times 2$ matrix $A$ and the vector $y \in \mathbb{R}^3$.)

(b) Use Matlab to compute the singular value decomposition $A = UDV^T$.

(c) Using the result of (b), construct $D^+$ and the minimum norm solution $x^+ := VD^+U^Ty$ in Matlab. Draw the three lines specified by the equations and the point $x^+$ in the $(x_1, x_2)$-plane. Discuss the result.

M2. The *condition number* of a square matrix $A$ of size $n \times n$ is defined by

$$\text{cond}(A) := \frac{d_1}{d_n},$$

where $d_1$ and $d_n$ are the first and last singular values of $A$, respectively.

Download the Matlab routine `XR01_buildA.m` from the course website. There, you can choose two crucial numbers: $N$ determines the size of the reconstruction (which is $N \times N$), and $T$ is the number of projection directions evenly distributed between 0 and 180 degrees.

(a) Let $T$ be constant and increase $N$ step by step. Use the command `spy(A)` to see the structure of the measurement matrix. What happens to $\text{cond}(A)$ when $N$ grows? Why?

(b) Let $N$ be constant and increase $T$ step by step. Use the command `spy(A)` to see the structure of the measurement matrix. What happens to $\text{cond}(A)$ when $T$ grows? Why?