Mathematical derivation of the Filtered Back-Projection method

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1 Radon transform and radiographs

As explained above, the starting point of X-ray tomography is the knowledge of line integrals of the unknown attenuation coefficient for a collection of lines. These lines are in three-dimensional space, but since sometimes it is convenient to measure 2-D slices of the object, we present measurement geometries in both $\mathbb{R}^2$ and $\mathbb{R}^3$.

Let us discuss the 2-D case first. Let $f(x) = f(x_1, x_2)$ be the attenuation coefficient. The most classical model for the data is the so-called *Radon transform*

$$Rf(\theta, s) = \int_{x\cdot\theta=s} f(x)dx = \int_{y\in\theta^\perp} f(s\theta + y)dy, \qquad \theta \in S^1, s \in \mathbb{R}, \quad (1)$$

where S^1 is the unit circle, $\theta^\perp$ is the orthogonal complement of the unit vector θ and $x \cdot \theta$ denotes vector inner product. We will abuse notation and let θ mean the unit vector $(\cos\theta, \sin\theta) \in \mathbb{R}^2$ parametrized by the angle $\theta \in [0, 2\pi]$.

An equivalent operator, intuitively better suited for X-ray tomography is the *parallel beam radiograph*

$$Pf : \{(\theta, x) \in S^1 \times \mathbb{R} \,|\, x \in \theta^\perp\} \to \mathbb{R}, \quad (2)$$

$$P_\theta f(s) = \int_{-\infty}^{\infty} f(x + t\theta)dt. \quad (3)$$

Note that here the unit vector θ points in the direction of the X-ray whereas in Radon transform they are orthogonal. First generation CT scanners were based on the parallel beam measurement geometry: with a fixed angle a collection of very thin, parallel rays were measured. As the angle varied over a half-circle, the whole parallel beam radiograph was achieved for a 2-D slice of the patient.

The need to lower patient dose suggests the use of a 2-D fan beam. Here we introduce the measurement circle A with radius R:

$$A = \{x \in \mathbb{R}^2 \,|\, |x| = R\}.$$

The *divergent beam radiograph* is given by

$$\mathcal{D}_a f(\theta) = \int_0^\infty f(a + t\theta)dt, \tag{4}$$

and we think of the X-ray source being located on A and sending a beam to direction θ.

The two radiographs are related by the formula

$$P_\theta f(E_\theta x) = D_x f(\theta) + D_x f(-\theta), \tag{5}$$

where

$$E_\theta(x) = x - (x \cdot \theta)\theta \tag{6}$$

is the orthogonal projection to the orthogonal complement $\theta^\perp$ of θ.

The 3-D version of Radon transform integrates over hyperplanes $x \cdot \theta = s$ and thus is not practically so useful as the two radiographs. They generalize to 3-D simply by replacing θ by a three-dimensional unit vector in the formulae. We remark that the 3-D version of the divergent beam radiograph is called the cone-beam transform.

2 Filtered Back-Projection

We present here the most popular CT algorithm called *filtered backprojection*. It is based on this basic idea: to reconstruct f at a point x, the most obvious data related to $f(x)$ are the integrals over lines passing through x. Let us sum them all together, call the result $Tf(x)$ and see what we get by introducing polar coordinates:

$$\begin{aligned} Tf(x) &= \int_0^\pi \int_{-\infty}^\infty f(x + t\theta)dtd\theta \\ &= \int_0^{2\pi} \int_0^\infty \frac{f(x + t\theta)}{t} tdtd\theta \\ &= \int_{\mathbb{R}^2} \frac{f(x + y)}{|y|} dy \\ &= \int_{\mathbb{R}^2} \frac{f(y)}{|x - y|} dy \\ &= (f(y) * \frac{1}{|y|})(x), \end{aligned} \tag{7}$$

where $*$ stands for convolution.

We want to find an inverse operator for T. Recall that Fourier transform converts convolution to multiplication (i.e. $\widehat{g*h} = \hat{g}\hat{h}$) and

$$\widehat{\frac{1}{|y|}}(\xi) = \frac{1}{|\xi|}.$$

Furthermore, define the Calderón operator Λ in all dimensions $\mathbb{R}^n$ by

$$\Lambda f(x) := \mathcal{F}^{-1}|\xi|\hat{f}(\xi) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}|\xi|\hat{f}(\xi)d\xi, \tag{8}$$

where $\mathcal{F}^{-1}$ is the inverse Fourier transform. Note that Λ can be thought of as a high-pass filter. Now we see that

$$\widehat{Tf}(\xi) = \frac{\hat{f}(\xi)}{|\xi|},$$

and thus

$$\Lambda T f = f. \tag{9}$$

On the other hand, we can relate Tf to the measurements with the following formula:

$$\begin{aligned} Tf(x) &= \int_0^\pi \int_{-\infty}^\infty f(E_\theta x + t\theta)dtd\theta \\ &= \int_0^\pi P_\theta f(E_\theta x)d\theta. \end{aligned}$$

Thus we arrive at the famous reconstruction formula

$$f(x) = \Lambda \int_0^\pi P_\theta f(E_\theta x)d\theta \tag{10}$$

originally proposed by Johann Radon in 1917.