

Inverse problems course exam

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The theoretical Problem 1 is mandatory for everyone. Problems 2–4 are computational, and you can choose freely which two of them to solve. In other words: you must do Problem 1, and any two out of the problem set {2,3,4}.

You are supposed to solve these problems, report your solutions, and write the necessary Matlab codes by yourself, not in teams. However, you can use all the Matlab codes in the course website and the book website as a basis of further modification.

Please return your solution report as an email attachment (in pdf format) sent to Andreas Hauptmann (Andreas.Hauptmann@helsinki.fi) at latest on Monday, March 16, 2015. Use the L^AT_EX file `2015_home_exam.tex`, namely the file creating this document, as a starting point of your report. Please send as well all the Matlab codes you wrote for the exam.

Here are some extra instructions to the computational problems 2–4. Please include plots of reconstructions, singular values, and other relevant numerical data. The vector-graphics image format “eps” is recommended especially for graphs, whereas images can be “eps” or pixel-based formats such as “jpg” or “png”. You might find these Matlab commands useful:

```
print -depsc image.eps  
print -dpng image.png
```

The idea in reporting your computational work is to make sure that the reader understands what was done and would be in principle capable of repeating your experiments.

Whenever relevant, please add a reference to the list in the end of the document. References are made for example like this: for more details, see [1] or [2, Chapter 7].

Firstname Lastname

Student number 00000000

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1 Theoretical basis of inversion

This is a mandatory task for everybody taking the exam.

We considered in the course indirect measurements $\tilde{m} = P_k \mathcal{A}f + \tilde{\varepsilon}$. Here f is a function defined on a subset of \mathbb{R}^d and $\tilde{m} \in \mathbb{R}^k$. The inverse problem is *given \tilde{m} , find f (or at least an approximation to f)*.

- (a) Formulate Hadamard's definition of a well-posed problem in the context of the above inverse problem. (See Figure 1 for a photo of Hadamard.)
- (b) Write down the definition of a *regularization strategy*. Draw a picture showing the data space, the parameter space, the direct map, \tilde{m} , the true solution f and the regularized inverse. Explain how the regularized solution can be understood as a noise-robust approximation of f .
- (c) Consider the one-dimensional deconvolution problem discussed in the course. Explain why that inverse problem is *ill-posed* in the sense of Hadamard. (Hint: the continuous forward map $f \mapsto \mathcal{A}f$ does not have a continuous inverse, which can be seen by using the Fourier transform). Include references to the literature or Internet resources you used.
- (d) We also considered a computational model $m = Af$ with a $k \times n$ matrix A . Explain the role of this model in practical inversion methods.



Figure 1: Photograph of Jacques Hadamard (1865–1963).

2 Inversion of the Laplace transform

2.1 Introduction

Definition and some applications of the Laplace transform. Explanation of what will be done in this study. Mention the ill-posedness of the inversion of the Laplace transform. Include references to literature whenever necessary.

Let $f : [0, \infty) \rightarrow \mathbb{R}$. The Laplace transform F of f is defined by

$$F(s) = \int_0^{\infty} e^{-st} f(t) dt, \quad s \in \mathbb{C}, \quad (1)$$

provided that the integral converges. The direct problem is to determine F for a given function f according to (1). The inverse problem is: *given a Laplace transform F , find the corresponding function f .*

2.2 Materials and Methods

2.2.1 The matrix model

Assume we know the values of F at these real-valued points:

$$0 < s_1 < s_2 < \dots < s_n < \infty.$$

Then we may approximate the integral in (1) for example with the trapezoidal rule as

$$\int_0^{\infty} e^{-st} f(t) dt \approx \frac{t_k}{k} \left(\frac{1}{2} e^{-s_1 t_1} f(t_1) + e^{-s_2 t_2} f(t_2) + e^{-s_3 t_3} f(t_3) + \dots \right. \\ \left. + e^{-s_{k-1} t_{k-1}} f(t_{k-1}) + \frac{1}{2} e^{-s_k t_k} f(t_k) \right), \quad (2)$$

where vector $t = [t_1 \ t_2 \ \dots \ t_k]^T \in \mathbb{R}^k$, $0 \leq t_1 < t_2 < \dots < t_k$, contains the points at which the unknown function f will be evaluated. By denoting $f_\ell = f(t_\ell)$, $\ell = 1, \dots, k$, and $m_j = F(s_j)$, $j = 1, \dots, n$, and using (2), we get a linear model of the form $m = Af + \epsilon$ with

$$A = \frac{t_k}{k} \begin{bmatrix} \frac{1}{2} e^{-s_1 t_1} & e^{-s_1 t_2} & e^{-s_1 t_3} & \dots & e^{-s_1 t_{k-1}} & \frac{1}{2} e^{-s_1 t_k} \\ \frac{1}{2} e^{-s_2 t_1} & e^{-s_2 t_2} & e^{-s_2 t_3} & \dots & e^{-s_2 t_{k-1}} & \frac{1}{2} e^{-s_2 t_k} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{2} e^{-s_n t_1} & e^{-s_n t_2} & e^{-s_n t_3} & \dots & e^{-s_n t_{k-1}} & \frac{1}{2} e^{-s_n t_k} \end{bmatrix}. \quad (3)$$

2.2.2 The inversion method

Give a brief explanation of the truncated SVD method.

2.3 Results

The Results section is for a detailed explanation of what happens when the methods of Section 4.2 are applied to the materials. There should be no interpretation of what the results might mean, just a dry and factual report with numbers, charts and plots.

- (a) Compute numerically and plot the Laplace transform of

$$f(t) = \begin{cases} t, & \text{for } 0 \leq t \leq 1/4, \\ 1/2 - t, & \text{for } 1/4 \leq t \leq 1/2, \\ 0, & \text{otherwise.} \end{cases}$$

- (b) Construct matrix A given by (3) for a suitable choice of points t_ℓ and s_j . Compute the singular values of A and include a plot of them here. Do you detect ill-posedness?
- (c) Use truncated SVD to compute the inverse Laplace transform of f . Choose the number of singular values so that it gives the smallest relative error in reconstruction (although this is kind of cheating: in actual practice we would not know the true solution).

2.4 Discussion

This section is for interpretations of the results presented in Section 2.3. Sometimes this section is called Conclusions, or Discussion and Conclusions.

3 Morozov discrepancy principle

3.1 Introduction

Explain the idea of Morozov's method for choosing the regularization parameter in Tikhonov regularization [2, Section 5.4.1]. Mention that the method will be tested here in the context of X-ray tomography.

3.2 Materials and Methods

3.2.1 The matrix model

Explain shortly the discretization of the two-dimensional parallel-beam X-ray tomography problem and the standard construction of the tomography matrix.

3.2.2 The Shepp-Logan phantom

Tell the reader about the Shepp-Logan phantom. Mention how inverse crime is avoided in your simulated data (it is fine to use the files developed in the course).

3.2.3 The inversion method

Give a brief explanation of Tikhonov regularization. Explain how Morozov's method can be numerically implemented.

3.3 Results

This section should contain a dry report about what happens when the above methods are applied to the above "materials," or simulated phantoms in this case.

Show how Morozov's method works for your numerical simulations.

3.4 Discussion

This section is for interpretations of the results presented in Section 3.3.

4 Backward heat equation

4.1 Introduction

Study the heat propagation model explained in [2, Section 2.2] and tell the reader about it here.

4.2 Materials and Methods

4.2.1 The matrix model

Explain shortly the discretization of the heat propagation [2, Section 2.2.2].

4.2.2 A discontinuous initial state

Choose an initial temperature distribution that is discontinuous. For example, in the interval $[0, 1]$ the heat distribution could be zero otherwise but 100°C in the interval $1/3 \leq t \leq 2/3$.

4.2.3 The inversion method

You can choose either total variation regularization or Haar wavelet inversion with a Besov space B_{11}^1 penalty. Use the sparsity-based parameter choice rule explained in [2, Section 6.3].

4.3 Results

This section should contain a dry report about what happens when the above methods are applied to the above “materials,” or simulated phantoms in this case.

Show a plot of the singular values of your measurement matrix. Further, show reconstructions of the discontinuous initial condition using the inversion method of your choice.

4.4 Discussion

This section is for interpretations of the results presented in Section 4.3.

References

- [1] K. HÄMÄLÄINEN, A. KALLONEN, V. KOLEHMAINEN, M. LASSAS, K. NIINIMÄKI, AND S. SILTANEN, *Sparse tomography*, SIAM Journal on Scientific Computing, 35 (2013), pp. B644–B665.
- [2] J. MUELLER AND S. SILTANEN, *Linear and Nonlinear Inverse Problems with Practical Applications*, vol. 10 of Computational Science and Engineering, SIAM, 2012.