Chapter 6: Cardinal Numbers and The Axiom of Choice

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Cardinal Arithmetic

Definition 6.4 (Addition)

Let κ and λ be two cardinal numbers. Define

$$\kappa + \lambda := |K \cup L|,$$

where K and L are any disjoint sets with $|K| = \kappa$ and $|L| = \lambda$.

Example 6.12: $\kappa + 0 = \kappa$ for any cardinal number κ.

Proof. Let K be any set with $|K| = \kappa$. Clearly, $|\emptyset| = 0$ and $K \cap \emptyset = \emptyset$.

We have that

$$\kappa + 0 = |K \cup \emptyset| = |K| = \kappa.$$

□

Theorem 6I

For any cardinal numbers κ, λ, μ:

1. $\kappa + \lambda = \lambda + \kappa$,
2. $\kappa + (\lambda + \mu) = (\kappa + \lambda) + \mu$.

Proof. Let K, L, M be pairwise disjoint sets such that $|K| = \kappa$, $|L| = \lambda$ and $|M| = \mu$. Clearly,

$$\kappa + \lambda = |K \cup L| = |L \cup K| = \lambda + \kappa,$$

and

$$\begin{aligned}\kappa + (\lambda + \mu) &= \kappa + |L \cup M| = |K \cup (L \cup M)| \\ &= |(K \cup L) \cup M| = |K \cup L| + \mu = (\kappa + \lambda) + \mu.\end{aligned}$$

□

Example 6.13: $n + \aleph_0 = \aleph_0$ for any natural number n.

Proof. Let $K = n \times \{0\}$. Clearly, $|K| = n$, $|\omega| = \aleph_0$ and $K \cap \omega = \emptyset$. It suffices to show that $(n \times \{0\}) \cup \omega \approx \omega$.

Claim: the function $f : (n \times \{0\}) \cup \omega \to \omega$ defined as follows is a bijection:

$$f(x) = \begin{cases} k, & \text{if } x = (k, 0) \text{ for some } k \in n, \\ n + x, & \text{if } x \in \omega. \end{cases}$$

Clearly, f is one-to-one. On the other hand, given any $m \in \omega$, we must find a pre-image for m under f. If $m \in n$, then $(m, 0) \in n \times \{0\}$ and $f(m, 0) = m$; if $m = n + x$ for some $x \in \omega$, then $f(x) = n + x = m$. □

Note: From this example, we know that cancellation law does not hold for addition on cardinals, as e.g., $2 + \aleph_0 = \aleph_0 = 3 + \aleph_0$, but $2 \neq 3$.

Example 6.14: $\aleph_0 + \aleph_0 = \aleph_0$.

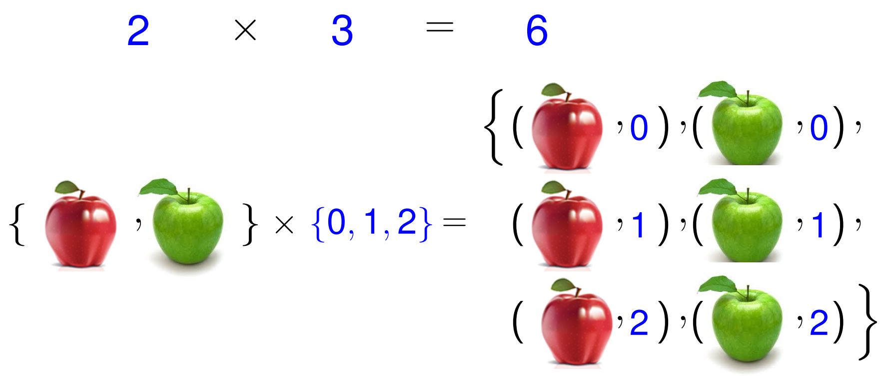
Proof. Clearly, $|\omega \times \{0\}| = |\omega \times \{1\}| = |\omega| = \aleph_0$. It suffices to show that $(\omega \times \{0\}) \cup (\omega \times \{1\}) \approx \omega$.

Clearly, the function $f : (\omega \times \{0\}) \cup (\omega \times \{1\}) \to \omega$ defined as follows is a bijection:

$$f(n, 0) = 2n,$$
$$f(n, 1) = 2n + 1.$$

□

Corollary: $\underbrace{\aleph_0 + \cdots + \aleph_0}_{n \text{ times}} = \aleph_0$ for any $n \in \omega$.



Definition 6.5 (Multiplication)

Let κ and λ be two cardinal numbers. Define

$$\kappa \cdot \lambda := |K \times L|,$$

where K and L are any sets with $|K| = \kappa$ and $|L| = \lambda$.

Example 6.15: For any cardinal κ,

$$\kappa \cdot 0 = 0 \quad \text{and} \quad \kappa \cdot 1 = \kappa.$$

Proof. Let K be any set such that $|K| = \kappa$. Clearly, $|\emptyset| = 0$ and $|1| = |\{\emptyset\}| = 1$.

We have that

$$\kappa \cdot 0 = |K \times \emptyset| = |\emptyset| = 0,$$

and that

$$\kappa \cdot 1 = |K \times \{\emptyset\}| = |K| = \kappa.$$

since $K \times \{\emptyset\} \approx K$. □

Theorem 6H

If $K_1 \approx K_2$ and $L_1 \approx L_2$, then $K_1 \times L_1 \approx K_2 \times L_2$.

Proof. Since $K_1 \approx K_2$ and $L_1 \approx L_2$, there are bijections $f : K_1 \to K_2$ and $g : L_1 \to L_2$. We show that the function $h : K_1 \times L_1 \to K_2 \times L_2$ defined as follows is a bijection:

$$h(x, y) = (f(x), g(y)).$$

For any $(x_1, y_1), (x_2, y_2) \in K_1 \times L_1$ such that $h(x_1, y_1) = h(x_2, y_2)$, we have that

$$\begin{aligned}(f(x_1), g(y_1)) = (f(x_2), g(y_2)) &\implies f(x_1) = f(x_2) \text{ and } g(y_1) = g(y_2)\\ &\implies x_1 = x_2 \text{ and } y_1 = y_2 \text{ (since } f \text{ and } g \text{ are 1-1)}\\ &\implies (x_1, y_1) = (x_2, y_2).\end{aligned}$$

For any $(w, v) \in K_2 \times L_2$, since f and g are surjective, there are $x \in K_1$ and $y \in L_1$ such that

$$f(x) = w \text{ and } g(y) = v.$$

Thus $(x, y) \in K_1 \times L_1$ and $h(x, y) = (f(x), g(y)) = (w, v)$. □

Example 6.16: For any nonzero natural number n,

$$n \cdot \aleph_0 = \aleph_0.$$

Proof. It suffices to prove that $n \times \omega \approx \omega$. Define a function $f : n \times \omega \to \omega$ by taking for all $m \in n$ and $k \in \omega$

$$f(m, k) = nk + m.$$

Note that by Division Algorithm, for every $p \in \omega$, there exist unique k, m such that $p = nk + m$ and $m \in n$. Hence f is a bijection. □

Example 6.17: $\aleph_0 \cdot \aleph_0 = \aleph_0$.

Proof. We have proved that $\omega \times \omega \approx \omega$, thus

$$\aleph_0 \cdot \aleph_0 = |\omega \times \omega| = |\omega| = \aleph_0.$$

□

Corollary: $\aleph_0 \underbrace{\cdot \cdots \cdot}_{n \text{ times}} \aleph_0 = \aleph_0$ for any $n \in \omega$.

Example 6.18: $\kappa + \kappa = 2 \cdot \kappa$.

Proof. Let K be a set such that $|K| = \kappa$. It suffices to prove that

$$(\{0\} \times K) \cup (\{1\} \times K) \approx 2 \times K.$$

But this is true since

$$(\{0\} \times K) \cup (\{1\} \times K) = (\{0\} \cup \{1\}) \times K = \{0, 1\} \times K = 2 \times K.$$

□

Analogously, we have $\kappa \underbrace{+ \cdots +}_{n \text{ times}} \kappa = n \cdot \kappa$ for any $n \in \omega$.

Theorem 6I

For any cardinal numbers κ, λ, μ:

1. $\kappa \cdot \lambda = \lambda \cdot \kappa$,
2. $\kappa \cdot (\lambda \cdot \mu) = (\kappa \cdot \lambda) \cdot \mu$.
3. $\kappa \cdot (\lambda + \mu) = \kappa \cdot \lambda + \kappa \cdot \mu$.

Proof. We only prove item 3. Let K, L, M be pairwise disjoint sets such that $|K| = \kappa$, $|L| = \lambda$ and $|M| = \mu$. We have that

$$\begin{aligned}\kappa \cdot (\lambda + \mu) &= \kappa \cdot |L \cup M| = |K \times (L \cup M)| \\ &= |(K \times L) \cup (K \times M)| \\ &= |(K \times L)| + |(K \times M)| \\ &= (\kappa \cdot \lambda) + (\kappa \cdot \mu).\end{aligned}$$

□

$2^3 = 2 \times 2 \times 2$

$$2^{\aleph_0} \quad : \quad \underbrace{2 \times \cdots \times 2}_{\omega \text{ times}} = \prod_{n \in \omega} 2 = {}^{\omega}2$$

Recall: ${}^{L}K = \{f : L \to K \mid f \text{ is a function}\}$.

Definition 6.6 (Exponentiation)

Let κ and λ be two cardinal numbers. Define

$$\kappa^{\lambda} := |{}^{L}K|,$$

where K and L are any sets with $|K| = \kappa$ and $|L| = \lambda$.

Example 6.19:

- $\kappa^1 = \kappa$ and $\kappa^0 = 1$, for any cardinal κ.
- $0^{\kappa} = 0$ for any nonzero cardinal κ.

In particular, $0^0 = 1$.

Proof. Let K be any set such that $|K| = \kappa$. We have that

$$\kappa^1 = |{}^{\{\emptyset\}}K| = |\{f : \{\emptyset\} \to K \mid f \text{ is a function}\}| = |K| = \kappa,$$

$$\kappa^0 = |{}^{\emptyset}K| = |\{f : \emptyset \to K \mid f \text{ is a function}\}| = |\{\emptyset\}| = 1,$$

and

$$0^{\kappa} = |{}^{K}\emptyset| = |\{f : K \to \emptyset \mid f \text{ is a function}\}| = |\emptyset| = 0.$$

□

Example 6.20: For any set A,

$$|\wp A| = 2^{|A|}.$$

In particular, $|\wp\omega| = 2^{\aleph_0}$.

Proof. We have shown that for any set A,

$$\wp A \approx {}^{A}2.$$

It then follows that $|\wp A| = |{}^{A}2| = 2^{|A|}$. □

By Cantor's Theorem, $|A| \neq |\wp A|$, thus $\aleph_0 \neq 2^{\aleph_0}$.

Theorem 6H

If $K_1 \approx K_2$ and $L_1 \approx L_2$, then ${}^{L_1}K_1 \approx {}^{L_2}K_2$.

Proof. Since $K_1 \approx K_2$ and $L_1 \approx L_2$, there are bijections $f : K_1 \to K_2$ and $g : L_1 \to L_2$. We show that the function $H : {}^{L_1}K_1 \to {}^{L_2}K_2$ defined by taking

$$H(j) = f \circ j \circ g^{-1}.$$

is a bijection.

- H is surjective, since for any $u \in {}^{L_2}K_2$, there is $j_u = f^{-1} \circ u \circ g \in {}^{L_1}K_1$ s.t.

$$H(j_u) = f \circ j_u \circ g^{-1} = f \circ (f^{-1} \circ u \circ g) \circ g^{-1} = \mathrm{id}_{K_2} \circ u \circ \mathrm{id}_{L_2} = u.$$

- H is injective, since

$$H(i) = H(j) \Longrightarrow f \circ i \circ g^{-1} = f \circ j \circ g^{-1} \Longrightarrow f^{-1} \circ (f \circ i \circ g^{-1}) \circ g = f^{-1} \circ (f \circ j \circ g^{-1}) \circ g,$$

which gives $i = j$. □

Theorem 6I

For any cardinal numbers κ, λ, μ:

4. $\kappa^{\lambda+\mu} = \kappa^\lambda \cdot \kappa^\mu$,
5. $(\kappa \cdot \lambda)^\mu = \kappa^\mu \cdot \lambda^\mu$.
6. $(\kappa^\lambda)^\mu = \kappa^{\lambda \cdot \mu}$.

Proof. We only prove item 6. Let K, L, M be sets such that $|K| = \kappa$, $|L| = \lambda$ and $|M| = \mu$. It suffices to show that

$${}^M({}^L K) \approx {}^{(L \times M)}K.$$

Define a function $H : {}^M({}^L K) \to {}^{(L \times M)}K$ by taking for all $f : M \to {}^L K$,

$$H(f) = g_f,$$

where $g_f : L \times M \to K$ is defined as $g_f(l, m) = f(m)(l)$. We show that H is a bijection.

To see that H is surjective, for any $g : L \times M \to K$, let $f : M \to {}^L K$ be defined as $f(m) = h$, where $h : L \to K$ is defined as $h(l) = g(l, m)$. We show that $H(f) = g$.

We have that for all $(l, m) \in L \times M$, $H(f)(l, m) = f(m)(l) = g(l, m)$, which means $H(f) = g$, as desired.

To see that H is injective, for any $f_1, f_2 \in {}^M({}^L K)$, we have that

$$\begin{aligned} H(f_1) = H(f_2) &\Longrightarrow H(f_1)(l, m) = H(f_2)(l, m) \text{ for all } (l, m) \in L \times M \\ &\Longrightarrow f_1(m)(l) = f_2(m)(l) \text{ for all } (l, m) \in L \times M \\ &\Longrightarrow f_1(m) = f_2(m) \text{ for all } m \in M \\ &\Longrightarrow f_1 = f_2. \end{aligned}$$

□

Theorem 6J

Let $m, n \in \omega$. Then

$$m + n = m +_\omega n.$$
$$m \cdot n = m \cdot_\omega n,$$
$$m^n = m^n,$$

where the exponentiation operation on the right hand side is the operation on ω.

Proof. Omitted. □

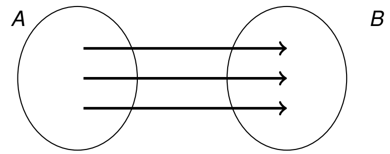
Theorem 6K

If A and B are finite, then $A \cup B$, $A \times B$ and ${}^B A$ are also finite.

Proof. By Theorem 6J. □

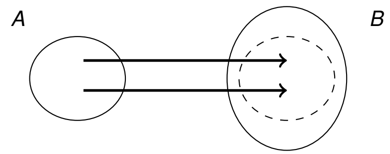
Ordering Cardinal Numbers

Two sets A and B are equinumerous iff there is a bijection $f : A \to B$.



Definition 6.7

A set A is dominated by a set B (written $A \preccurlyeq B$) iff there is a one-to-one function from A into B.



Note:

- If $A \subseteq B$, then $A \preccurlyeq B$, as the inclusion map $\iota : A \to B$, defined as $\iota(x) = x$, is a one-to-one function.
- If $A \approx B$, then $A \preccurlyeq B$. In particular, $A \preccurlyeq A$.
- If $A \preccurlyeq B$, then $A \approx C$ for some set $C \subseteq B$.

Definition 6.8

Let κ and λ be cardinal numbers. Define

$$\kappa \leq \lambda \quad \text{iff} \quad K \preccurlyeq L,$$

where K and L are two sets with $|K| = \kappa$ and $|L| = \lambda$.

For example:

- If $K \preccurlyeq L$, then $|K| \leq |L|$.
- $0 \leq \kappa$ for all cardinal κ, as the empty function $\emptyset : \emptyset \to K$ is one-to-one, where $|K| = \kappa$.

We need to check the above definition is independent of the choice of the sets K and L.

Theorem 6.9

If $K_1 \approx K_2$ and $L_1 \approx L_2$, then $K_1 \preccurlyeq L_1$ iff $K_2 \preccurlyeq L_2$.

Proof. Since $K_1 \approx K_2$ and $L_1 \approx L_2$, there are bijections $f : K_2 \to K_1$ and $g : L_1 \to L_2$.

Suppose $K_1 \preccurlyeq L_1$ and $h : K_1 \to L_1$ is a one-to-one function. It is easy to check that $g \circ h \circ f : K_2 \to L_2$ is a one-to-one function, thus $K_2 \preccurlyeq L_2$. Analogously, $K_2 \preccurlyeq L_2 \implies K_1 \preccurlyeq L_1$. □

Definition 6.10

Let κ and λ be cardinal numbers. Define

$$\kappa < \lambda \quad \text{iff} \quad \kappa \leq \lambda \text{ and } \kappa \neq \lambda.$$

Thus:

- $|K| < |L|$ iff $K \preccurlyeq L$ and $K \not\approx L$.
- $\kappa \leq \lambda$ iff either $\kappa < \lambda$ or $\kappa = \lambda$.

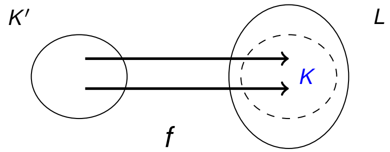
Note: $K \subset L \not\Longrightarrow |K| < |L|$, since, e.g., the set $\omega \setminus \{0\}$ is a proper subset of ω, but $|\omega \setminus \{0\}| = |\omega|$ instead of $|\omega \setminus \{0\}| < |\omega|$.

Example 6.21:

(1) If $A \subseteq B$, then $|A| \leq |B|$.

(2) If $\kappa \leq \lambda$, then there are sets K and L such that $K \subseteq L$, $|K| = \kappa$ and $|L| = \lambda$.

Proof. (of (2)) Suppose K' and L are sets such that $|K'| = \kappa$ and $|L| = \lambda$. Thus $K' \preccurlyeq L$ and there is a one-to-one function $f : K' \to L$. Let $K = \operatorname{ran} f$. Then $K \subseteq L$ and $K \approx K'$, thereby $|K| = |K'| = \kappa$.



Example 6.22:

(1) $n < \aleph_0$ for any finite cardinal n.

(2) For any two finite cardinals m and n,

$$m \underline{\in} n \iff m \leq n.$$

That is, the ordering $\underline{\in}$ of Chapter 4 coincides with the cardinal ordering $\leq$.

Proof. (1) $n \subset \omega$ implies $n \leq \aleph_0$. $n \neq \omega$ since $n \not\approx \omega$.

(2) "$\Longrightarrow$": Clearly, $m \underline{\in} n \Longrightarrow m \subseteq n \Longrightarrow m \leq n$. "$\Longleftarrow$": If $m \leq n$, then there is a one-to-one function $f : m \to n$, thus $m \approx \operatorname{ran} f \subseteq n$. If $n \in m$, then $m \approx \operatorname{ran} f \subseteq n \subset m$, thus m is equinumerous to a proper subset of m, which contradicts the Pigeonhole Principle. Hence the trichotomy of $\in$ gives that $m \underline{\in} n$, as desired. □

Example 6.23: $\kappa < 2^\kappa$ for any cardinal κ.

Proof. Let K be a set such that $|K| = \kappa$. Then $|\wp K| = 2^{|K|} = 2^\kappa$.

The function $f : K \to \wp K$ defined by taking

$$f(x) = \{x\}$$

is a one-to-one function, thus $K \preccurlyeq \wp K$, i.e., $\kappa \leq 2^\kappa$.

On the other hand, by Cantor's Theorem, $|K| \neq |\wp K|$, thus $\kappa \neq 2^\kappa$.

In conclusion, $\kappa < 2^\kappa$. □

Example 6.24: $\kappa \leq \lambda \leq \mu \Longrightarrow \kappa \leq \mu$

Proof. Suppose $\kappa \leq \lambda \leq \mu$. Let K, L and M be sets such that $|K| = \kappa$, $|L| = \lambda$ and $|M| = \mu$. Then there exist one-to-one functions $f : K \to L$ and $g : L \to M$, which implies that $g \circ f : K \to M$ is a one-to-one function. Thus $K \preccurlyeq L$, i.e., $\kappa \leq \lambda$. □

Properties of $\leq$:

1. $\kappa \leq \kappa$.
2. $\kappa \leq \lambda \leq \mu \Longrightarrow \kappa \leq \mu$.
3. $\kappa \leq \lambda$ and $\lambda \leq \kappa \Longrightarrow \kappa = \lambda$. (Cantor-Schröder-Bernstein Theorem)
4. Either $\kappa \leq \lambda$ or $\lambda \leq \kappa$. (By Axiom of Choice)

Cantor-Schröder-Bernstein Theorem

(a) *If $A \preccurlyeq B$ and $B \preccurlyeq A$, then $A \approx B$.*

(b) *For any cardinals κ and λ, $\kappa \le \lambda \le \kappa \Longrightarrow \kappa = \lambda$.*

Proof. Postponed. □

Applications of the theorem:

Example 6.25: If $A \subseteq B \subseteq C$ and $A \approx C$, then $A \approx B \approx C$.

Proof. By assumption $|A| = |C| = \kappa$ and $|B| = \lambda$ for some cardinals κ, λ. Moreover, $\kappa \le \lambda \le \kappa$. It follows from C-S-B Theorem that $\kappa = \lambda$, which implies $A \approx B \approx C$. □

Example 6.26: $\mathbb{R} \approx [0, 1]$.

Proof. By Example 6.7, we have $\mathbb{R} \approx (0, 1)$. Noting that

$$(0, 1) \subseteq [0, 1] \subseteq \mathbb{R},$$

we conclude $(0, 1) \approx [0, 1] \approx \mathbb{R}$ by Example 6.18. □

Example 6.27:

- $\kappa \le \lambda < \mu \Longrightarrow \kappa < \mu$
- $\kappa < \lambda \le \mu \Longrightarrow \kappa < \mu$

Proof. We have that

$$\kappa \le \lambda < \mu \Longrightarrow \kappa \le \lambda \le \mu \Longrightarrow \kappa \le \mu$$

by transitivity of $\le$.

Now, if $\kappa = \mu$, then the assumption implies $\kappa \le \lambda \le \kappa$, which by C-S-B Theorem gives $\kappa = \lambda = \mu$, contradicting $\lambda < \mu$. Hence we conclude $\kappa < \mu$.

$\kappa < \lambda \le \mu \Longrightarrow \kappa < \mu$ is proved symmetrically. □