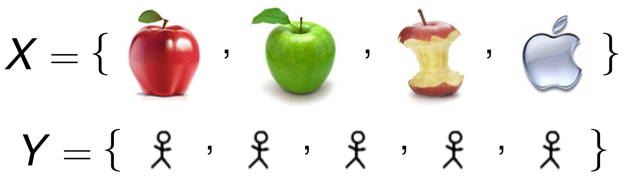
Chapter 6: Cardinal Numbers and The Axiom of Choice

March 18 & 20, 2014

Lecturer: Fan Yang

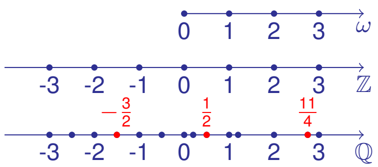
Equinumerosity

- What is the *size* of a set?

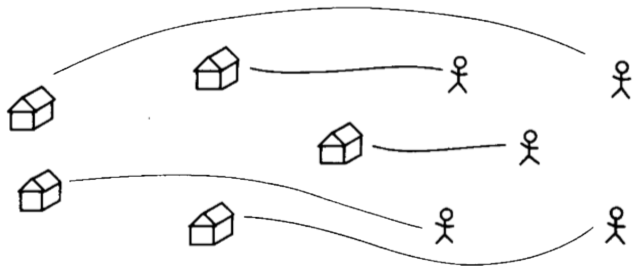


$\omega = \{0, 1, 2, 3, 4, \dots\}$

- Do A and B have the same size?
- Does A have more elements than B?
    - Y has more elements than X.
    - The infinite set ω has more elements than the finite sets X and Y.
    - Do ω and $\mathbb{Z}$ have the same size? What about $\mathbb{Q}$ and $\mathbb{Z}$?



    - Given two infinite sets A and B, how to compare their sizes?



Are there exactly as many houses as people?

- Yes, since there are 5 houses and 5 people.
- Yes, since there is a one-to-one correspondence between the two sets.

Definition 6.1

A set A is said to be equinumerous or equipotent to a set B (written $A \approx B$) iff there is a bijection from A onto B.

A bijection from A onto B is also called a *one-to-one correspondence* between sets A and B.

Are there exactly as many knives as forks?

Yes, as there is a one-to-one correspondence between knives and forks.

Example 6.1: Let $A = \{a, b, c, d\}$ and $B = \{a, b, c\}$. Then $A \not\approx B$, since there is no bijection from A onto B. In general, for any two finite sets X and Y, if $Y \subset X$, then $X \not\approx Y$.

Consider the following infinite sets. Clearly, $\mathbf{Even} \subset \omega$.

$$\begin{array}{rl} \omega = & \{0, 1, 2, 3, 4, 5, 6, \dots\}, \\ & \ \ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \searrow\ \searrow \\ \mathbf{Even} = & \{0, 2, 4, 6, 8, 10, 12, \dots\} = \{2n \mid n \in \omega\}. \end{array}$$

Example 6.2: $\omega \approx \mathbf{Even}$.

Proof. The function $f : \omega \to \mathbf{Even}$ defined by taking

$$f(n) = 2n$$

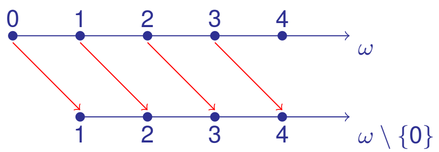
is a bijection. Indeed, f is injective, since $n \neq m \Longrightarrow 2n \neq 2m$.
f is surjective, since for all $m \in \mathbf{Even}$, $m = 2n$ for some $n \in \omega$, and $f(n) = 2n = m$. □

Example 6.3: [Galileo] Similarly, $\omega \approx \mathbf{Sq}$, where

$$\begin{array}{rl} \omega = & \{0, 1, 2, 3, 4, 5, \dots\}, \\ & \ \ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \searrow \\ \mathbf{Sq} = & \{0, 1, 4, 9, 16, 25, \dots\} = \{n^2 \mid n \in \omega\}. \end{array}$$

since there is a bijection $f : \omega \to \mathbf{Sq}$ defined as $f(n) = n^2$.

Example 6.4: $\omega \approx \omega \setminus \{0\}$, since the function $f : \omega \to \omega \setminus \{0\}$ defined as $f(n) = n^+$ is a bijection.



Remark: For infinite sets A, B,

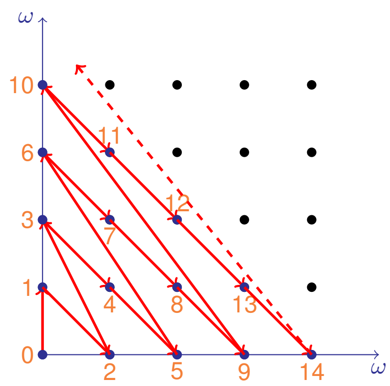
$$A \subset B \not\Longrightarrow A \not\approx B\,!$$

Example 6.5: $\omega \times \omega \approx \omega$.

Proof. The function $f : \omega \times \omega \to \omega$ defined as

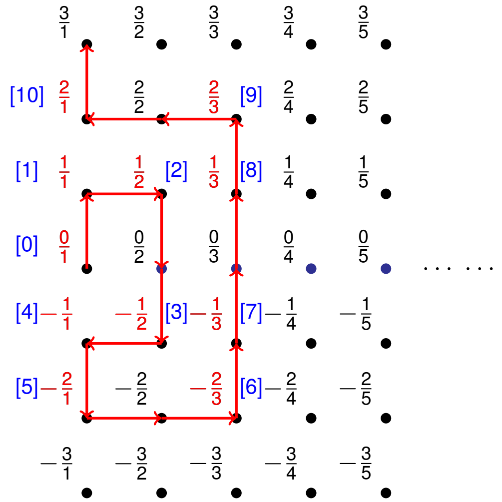
$$f(m, n) = \frac{1}{2}[(m + n)^2 + 3m + n]$$

is a bijection. □



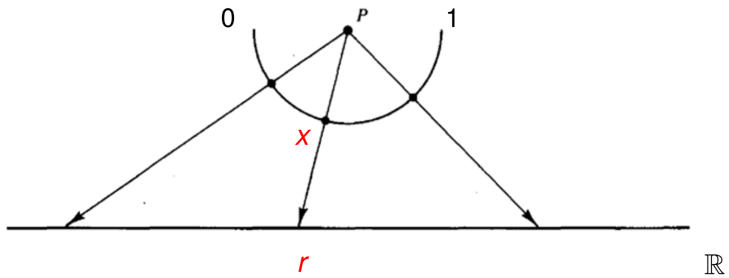
Example 6.6: $\mathbb{Q} \approx \omega$.

Proof. In the following picture, we specify a bijection $f : \omega \to \mathbb{Q}$. □



Example 6.7: $(0,1) \approx \mathbb{R}$, where $(0,1) = \{x \in \mathbb{R} \mid 0 < x < 1\}$.

Proof. By picture. Here $(0,1)$ has been bent into a semicircle with center P. Each point in $(0,1)$ is paired with its projection (from P) on the real line. □



An alternative proof: Claim: the mapping $f : (0,1) \to \mathbb{R}$ defined by putting

$$f(x) = \frac{1}{x} - \frac{1}{1-x}$$

is a bijection.

For any $a, b \in (0,1)$, if $f(a) = f(b)$, namely, if $\frac{1}{a} - \frac{1}{1-a} = \frac{1}{b} - \frac{1}{1-b}$, then $(b-a)\left(\frac{1}{ab} + \frac{1}{(1-a)(1-b)}\right) = 0$ thereby $a = b$. Hence f is injective.

For any $r \in \mathbb{R}$, there is $x_r = \frac{2}{r+\sqrt{4+r^2}+2} \in (0,1)$ such that

$$\begin{aligned} f(x_r) &= \frac{1}{x_r} - \frac{1}{1-x_r} = \left(\frac{1}{x_r} - 2\right)\frac{1}{1-x_r} \\ &= \frac{r+\sqrt{4+r^2}-2}{2} \cdot \frac{r+\sqrt{4+r^2}+2}{r+\sqrt{4+r^2}} \\ &= \frac{(r+\sqrt{4+r^2})^2 - 2^2}{2(r+\sqrt{4+r^2})} = r. \end{aligned}$$

Hence f is surjective. □

Recall : for any sets X and Y,

$$^{X}Y = \{f : X \to Y \mid f \text{ is a function}\}.$$

Example 6.8: For any set A, we have $\wp A \approx {}^{A}2$.

Proof. Define a function $H : \wp A \to {}^{A}2$ by taking

$$H(B) = f_B,$$

where $f_B : A \to \{0,1\}$ is the characteristic function of B defined as

$$f_B(x) = \begin{cases} 1 & \text{if } x \in B, \\ 0 & \text{if } x \in A \setminus B. \end{cases}$$

Claim: H is a bijection.

Indeed, H is injective, since for any $B, C \subseteq A$,

$$\begin{aligned} H(B) = H(C) &\Longrightarrow f_B = f_C \Longrightarrow \forall x \in A(f_B(x) = f_C(x)) \\ &\Longrightarrow \forall x \in A(x \in B \leftrightarrow x \in C) \Longrightarrow B = C; \end{aligned}$$

H is surjective, since for any function $g \in {}^{A}2$, there is $B = \{x \in A \mid g(x) = 1\} \subseteq A$ such that $H(B) = f_B = g$. □

Theorem 6A

For any sets A, B and C:

(a) $A \approx A$.

(b) If $A \approx B$, then $B \approx A$.

(c) If $A \approx B$ and $B \approx C$, then $A \approx C$.

Proof. Exercise. □

That is, $\approx$ is an "equivalence relation" on the class of all sets.

Theorem 6B (Cantor 1873)

(a) *The set ω is not equinumerous to the set $\mathbb{R}$ of real numbers.*
(b) *No set is equinumerous to its power set.*

Proof. (a) [Diagonal argument] Given any map $f : \omega \to \mathbb{R}$. We show that there exists $z \in \mathbb{R}$ such that $z \notin \operatorname{ran} f$.

$$f(0) = 236.\ 0012\ldots$$
$$f(1) = -7.\ 7374\ldots$$
$$f(2) = 3.\ 1415\ldots$$
$$f(3) = 0.\ 5246\ldots$$
$$\vdots$$

The integer part of z is 0, and the $(n+1)$st decimal place of z is 3 unless the $(n+1)$st decimal place of $f(n)$ is 3, in which case the $(n+1)$st decimal place of z is 4. For example, in the case shown,

$$z = 0.3433\ldots$$

Clearly, $z \neq f(n)$ for all n, as it differs from $f(n)$ in the $(n+1)$st decimal place. Hence $z \notin \operatorname{ran} f$.

(b) Suppose $g : A \to \wp A$ is surjective. Let

$$B = \{x \in A \mid x \notin g(x)\}.$$

Since $B \in \wp A$ and g is surjective, there exists $x_0 \in A$ such that

$$g(x_0) = B.$$

But then, by the definition of B,

$$x_0 \in B \iff x_0 \notin g(x_0) = B,$$

which is a contradiction. Hence there is no surjection from A onto $\wp A$, thus $A \not\approx \wp A$. □

Remark: We will soon be able to prove $\mathbb{R} \approx \wp\omega$.
$\mathbb{R}$ has smaller size than $\wp\mathbb{R}$, which has smaller size than $\wp\wp\mathbb{R}$, etc.

Finite Sets

There are finitely many forks in the picture. Why?

Because there are exactly 6 forks. This, in turn, is because there is a one-to-one correspondence between the natural number $6 = \{0, 1, 2, 3, 4, 5\}$ and the set F of forks, or in other words, $6 \approx F$.

Definition 6.2

A set is finite iff it is equinumerous to some natural number. Otherwise it is infinite.

That is, for any set A:

- A is finite iff $\exists n \in \omega (A \approx n)$
- A is infinite iff $\forall n \in \omega (A \not\approx n)$

Example 6.9: The set $A = \{a_0, a_1, a_2, a_3\}$ is finite, since $A \approx 4$ via the bijection $f : 4 \to A$ defined as: $f(n) = a_n$.

$$4 = \{0, 1, 2, 3\}$$
$$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow$$
$$A = \{a_0, a_1, a_2, a_3\}$$

Example 6.10: Any finite set A is not equinumerous to an infinite set B.

Proof. Suppose $A \approx B$. As A is finite, $A \approx n$ for some $n \in \omega$. From the transitivity of $\approx$, it follows that $B \approx n$, contradicting B being infinite. □

Next, we want to check that each finite set A is equinumerous to a *unique* natural number n. This requires the Pigeonhole Principle.

Pigeonhole Principle:

If n items are put into m pigeonholes with $n > m$, then at least one pigeonhole must contain more than one item.

Here $n = 10$ and $m = 9$.

Pigeonhole Princinple

No natural number is equinumerous to a proper subset of itself.

$$4 = \{0, 1, 2, 3\} \quad \text{“Pigeons”}$$
$$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow$$
$$x = \{3, 1, 0\} \quad \text{“Pigeonholes”}$$

In particular, $n \not\approx m$ for any $m \in n$.

Proof. (of Pigeonhole Principle) We shall show that for any $n \in \omega$, if $f : n \to n$ is a one-to-one function, then $\operatorname{ran} f = n$ (not a proper subset of n), namely, that the set T is inductive:

$$T = \{n \in \omega \mid \operatorname{ran} f = n \text{ for any one-to-one function } f : n \to n\}.$$

Base case: $0 \in T$, since the only function $f : 0 \to 0$ is the empty function $f = \emptyset$, and $\operatorname{ran} \emptyset = \emptyset$.

Inductive step: Assume $k \in T$ and $f : k^+ \to k^+$ is a one-to-one function. We show that $\operatorname{ran} f = k^+$. Note that $f \restriction k$ is a one-to-one function from k into k^+.

Case 1: $\operatorname{ran} (f \restriction k) \subseteq k$. Then $f \restriction k : k \to k$ is a one-to-one function, as $f : k^+ \to k^+$ is 1-1. Thus, the assumption $k \in T$ implies $\operatorname{ran} f \restriction k = k$. Moreover, since f is 1-1, we must have $f(k) = k$. Hence $\operatorname{ran} f = k \cup \{k\} = k^+$.

Case 2: $\operatorname{ran} (f \restriction k) \not\subseteq k$. Then $f(p) = k$ for some $p \in k$. Define $\hat{f} : k^+ \to k^+$ as

$$\hat{f}(p) = f(k),$$
$$\hat{f}(k) = f(p) = k,$$
$$\hat{f}(x) = f(x) \text{ for other } x \in k^+.$$

Then $\hat{f} : k^+ \to k^+$ is one-to-one, and $\hat{f}(k) = k$. By Case 1,

$$k^+ = \operatorname{ran} \hat{f} = \operatorname{ran} f.$$

□

Corollary 6C

No finite set is equinumerous to a proper subset of itself.

Proof. Let A be an arbitrary finite set. Suppose $f : A \to B$ is a bijection for some $B \subset A$. Let $g : A \to n$ be a bijection for some $n \in \omega$.

The function $g \circ f \circ g^{-1}$ is a bijection from n onto a proper subset $g[B]$ of n (see picture on blackboard), contradicting the Pigeonhole Principle. □

Corollary 6D

(a) Any set equinumerous to a proper subset of itself is infinite.
(b) The set ω is infinite.

Proof. (a) follows immediately from Corollary 6C.
(b) follows from (a), since e.g. $\omega \approx \omega \setminus \{0\}$, via the bijection $f : \omega \to \omega \setminus \{0\}$ defined as $f(n) = n^+$. □

Corollary 6E

Any finite set A is equinumerous to a unique natural number n, called the cardinal number or the cardinality of A, denoted by $\text{card}\, A$ or $|A|$.

Proof. Assume that $A \approx m$ and $A \approx n$ for some natural numbers m, n. It follows that $m \approx n$. Thus $m \not\in n$ and $n \not\in m$, which by trichotomy and Corollary 4M implies that $m = n$ is the case. □

It follows from the corollary that $A \approx n \iff |A| = n$.

For example, since $n \approx n$, we have $|n| = n$. If a, b, c, d are distinct objects, then $|\{a, b, c, d\}| = 4$, as $\{a, b, c, d\} \approx \{0, 1, 2, 3\}$.

Clearly, for finite sets A, B:

$$|A| = |B| \iff (|A| = n \wedge |B| = n) \iff (A \approx n \wedge B \approx n) \iff A \approx B.$$

We also want to have cardinal numbers for infinite sets. In fact, what sets these "numbers" are is not too crucial, but the essential demand is that we will define the cardinality $|A|$ for arbitrary set A in such a way that

$$|A| = |B| \iff A \approx B$$

is the case. We postpone until Chapter 7 the actual definition of the set $|A|$. The information we need for the present chapter is embodied in the following promise:

Promise: For any set A, we will define a set $|A|$ in such a way that:

1. $|A| = |B| \iff A \approx B$,
2. for a finite set A, the cardinal number $|A|$ is the natural number n for which $A \approx n$.

Definition 6.3

A cardinal number κ is the cardinality of some set, i.e., κ is a cardinal number iff $\kappa = |A|$ for some set A.

For example,

- Every natural number n is a cardinal number, as $n = |n|$.
- $|\omega|$ is a cardinal number, and name (due to Cantor) $|\omega| = \aleph_0$.

There is a unique set whose cardinality is 0, namely the empty set $\emptyset$.

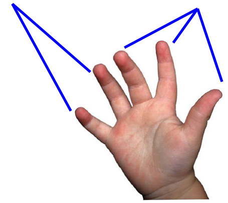
Given a cardinal number $\kappa > 0$, there are many sets A of cardinality κ. If one set A of cardinality κ is a finite set, then all sets of cardinality κ are finite sets. In this case, κ is called a finite cardinal. Otherwise, κ is called an infinite cardinal.

- Natural numbers are finite cardinals, and they are the only finite cardinals.
- $\aleph_0$, $|\mathbb{R}|$, $|\wp\omega|$, $|\wp\wp\omega|$ are infinite cardinals. Note $\aleph_0 \neq |\mathbb{R}|$ and $\aleph_0 \neq |\wp\omega| \neq |\wp\wp\omega|$ as we have shown $\omega \not\approx \mathbb{R}$, and $\omega \not\approx \wp\omega \not\approx \wp\wp\omega$.

Cardinal Arithmetic

Elementary school math:

$$2 \quad + \quad 3 \quad = \quad 5$$



Definition 6.4 (Addition)

Let κ and λ be two cardinal numbers. Define

$$\kappa + \lambda := |K \cup L|,$$

where K and L are any disjoint sets with $|K| = \kappa$ and $|L| = \lambda$.

Note: It is always possible to take disjoint sets K, L in the above definition, as $K \times \{0\}$ and $L \times \{1\}$ are always disjoint.

Example 6.11: Prove: $2 + 2 = 4$.

Proof. Let $K = 2 \times \{0\}$ and $L = 2 \times \{1\}$. Clearly, $K \cap L = \emptyset$ and $|K| = 2 = |L|$. We need to show that $K \cup L \approx 4$.

We have that

$$K \cup L = (2 \times \{0\}) \cup (2 \times \{1\}) = \{(0,0), (1,0), (0,1), (1,1)\}.$$

Clearly, the function $f : 4 \to K \cup L$ defined by taking

$$f(0) = (0,0), \quad f(1) = (1,0), \quad f(2) = (0,1), \quad f(3) = (1,1)$$

is a bijection. □

Theorem 6H

Assume that $K_1 \approx K_2$ and $L_1 \approx L_2$. If $K_1 \cap L_1 = K_2 \cap L_2 = \emptyset$, then $K_1 \cup L_1 \approx K_2 \cup L_2$.

Proof. Since $K_1 \approx K_2$ and $L_1 \approx L_2$, there are bijections $f : K_1 \to K_2$ and $g : L_1 \to L_2$. Define a function $h : K_1 \cup L_1 \to K_2 \cup L_2$ by taking

$$h(x) = \begin{cases} f(x), & \text{if } x \in K_1, \\ g(x), & \text{if } x \in L_1. \end{cases}$$

Since $K_1 \cap L_1 = \emptyset$, h is indeed a function. Claim: h is a bijection.

For any $x_1, x_2 \in K_1 \cup L_1$ such that $h(x_1) = h(x_2) = y$, since $K_2 \cap L_2 = \emptyset$, y is in exactly one set of K_2 and L_2. W.l.o.g., assume that $y \in K_2$. Then $x_1, x_2 \in K_1$. Since f is injective, $x_1 = x_2$. Hence h is injective.

We have that $h[K_1 \cup L_1] = h[K_1] \cup h[L_1] = K_2 \cup L_2$, since $h[K_1] = f[K_1] = K_2$ and $h[L_1] = g[L_1] = L_2$. Hence h is surjective. □