Chapter 5: Construction of the Real Numbers

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Real Numbers

Definition 5.14 (Dedekind Cut)

A Dedekind cut is a non-empty proper subset x of $\mathbb{Q}$ such that

(i) *x is closed downwards, i.e., for all $q, r \in \mathbb{Q}$,*
$$q \in x \text{ and } r < q \implies r \in x.$$

(ii) *x has no greatest element, i.e., there is no $a \in x$ such that $b \leq a$ for all $b \in x$.*

Definition 5.15

The set $\mathbb{R}$ of real numbers is the set of all Dedekind cuts.
That is, a real number is a Dedekind cut.

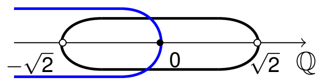
Example 5.11: Given any $q \in \mathbb{Q}$, the set $q_{\mathbb{R}} := \{r \in \mathbb{Q} \mid r < q\}$ is a Dedekind cut, thus a real number.

Proof.

- $q_{\mathbb{R}} \neq \emptyset$, since e.g. $q - 1 \in q_{\mathbb{R}}$.
- $q_{\mathbb{R}} \neq \mathbb{Q}$, since e.g. $q \notin q_{\mathbb{R}}$.
- For any $s, t \in \mathbb{Q}$, $\quad s < t \in q_{\mathbb{R}} \implies s < t < q \implies s \in q_{\mathbb{R}}$.
- For any $r \in q_{\mathbb{R}}$, we have $r < q$. Since $\mathbb{Q}$ is dense, there exists $s \in \mathbb{Q}$ such that $r < s < q$, thus $s \in q_{\mathbb{R}}$. □

In particular, $0_{\mathbb{R}} = \{r \in \mathbb{Q} \mid r < 0\}$ and $1_{\mathbb{R}} = \{r \in \mathbb{Q} \mid r < 1\}$ are real numbers.

Example 5.12: $\sqrt{2}_{\mathbb{R}} := \{r \in \mathbb{Q} \mid r \cdot r < 2 \vee r < 0\} = x$ is a real number.



Proof. Clearly, $x \neq \emptyset$, since e.g. $-1 \in x$. Since $2 \cdot 2 = 4 > 2$, we have $2 \notin x$, thus $x \neq \mathbb{Q}$.

x is closed downwards: Given any $q, r \in \mathbb{Q}$ such that $r < q \in x$.

- If $r < 0$, then $r \in x$ and we are done.
- If $r = 0$, then $r \cdot r = 0 \cdot 0 = 0 < 2$, which implies $r \in x$ and we are also done.
- Now assume that $0 < r < q$. Then it follows that
$$r \cdot r < q \cdot r < q \cdot q < 2,$$
which implies $r \in x$.

x has no greatest element: Given any $r \in x$, we need to find a $q \in x$ such that $r < q$.

- If $r < 0$, then let $q = 0$. Clearly $r < q$. Moreover, since $q \cdot q = 0 < 2$, we have $q \in x$.
- If $r \geq 0$, then let $q = \frac{2r+2}{r+2} = (2r+2)(r+2)^{-1} \in \mathbb{Q}$. We have that

$$r < \frac{2r+2}{r+2} \overset{r\geq 0}{\iff} r(r+2) < 2r+2 \iff r \cdot r < 2.$$

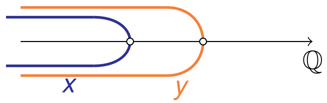
Moreover,

$$\begin{aligned} q \in x \Longleftarrow q \cdot q < 2 &\iff \left(\frac{2r+2}{r+2}\right)^2 < 2 \\ &\overset{r\geq 0}{\iff} (2r+2)^2 < 2(r+2)^2 \\ &\iff 4r^2 + 8r + 4 < 2r^2 + 8r + 8 \\ &\iff 2r^2 < 4 \\ &\iff r^2 < 2. \end{aligned}$$

Hence, since $r \cdot r < 2$, we obtain $r < q \in x$. □

Definition 5.16

Define an ordering $<_{\mathbb{R}}$ on $\mathbb{R}$ as: $x <_{\mathbb{R}} y$ iff $x \subset y$.



Define $x \leq_{\mathbb{R}} y$ iff $x <_{\mathbb{R}} y$ or $x = y$.

Example 5.13:

- $0_{\mathbb{R}} <_{\mathbb{R}} 1_{\mathbb{R}}$, since $\{r \in \mathbb{Q} \mid r < 0\} \subset \{r \in \mathbb{Q} \mid r < 1\}$.
- If p, q are two rationals with $p < q$, then $p_{\mathbb{R}} <_{\mathbb{R}} q_{\mathbb{R}}$.

  Proof. We need to show that

$$\{r \in \mathbb{Q} \mid r < p\} \subset \{r \in \mathbb{Q} \mid r < q\}.$$

  Clearly, "$\subseteq$" holds. To see $p_{\mathbb{R}} \neq q_{\mathbb{R}}$, observe that there exists $s \in \mathbb{Q}$ such that $p < s < q$, as $\mathbb{Q}$ is dense with respect to $<$. It follows that $s \notin p_{\mathbb{R}}$ and $s \in q_{\mathbb{R}}$, thereby $p_{\mathbb{R}} \neq q_{\mathbb{R}}$. □

Example 5.14: $1_{\mathbb{R}} <_{\mathbb{R}} \sqrt{2}_{\mathbb{R}}$.

Proof. We need to show that

$$\{r \in \mathbb{Q} \mid r < 1\} \subset \{r \in \mathbb{Q} \mid r \cdot r < 2 \vee r < 0\}.$$

Since $1 \in \sqrt{2}_{\mathbb{R}}$, but $1 \notin 1_{\mathbb{R}}$, we have $1_{\mathbb{R}} \neq \sqrt{2}_{\mathbb{R}}$.

For any $r \in 1_{\mathbb{R}}$, we have $r < 1$.

- If $r \leq 0$, then obviously $r \in \sqrt{2}_{\mathbb{R}}$.
- If $r > 0$, then $r < 1 \Longrightarrow r \cdot r < r < 1 < 2$, thus $r \in \sqrt{2}_{\mathbb{R}}$. □

Definition 5.17

Let $<$ be a linear ordering on a set A, and B a subset of A.
An element $a \in A$ is said to be an upper bound of B in A iff

$$b \leq a \text{ for all } b \in B.$$

The set B is said to be bounded (above) in A iff there exits some upper bound of B.
An element $a \in A$ is said to be a lower bound of B in A iff

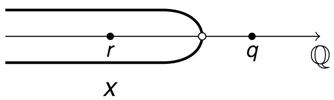
$$a \leq b \text{ for all } b \in B.$$

Note: An upper bound of B does not have to be an element of B. Similarly for lower bounds.

Example 5.15: Let $B = 1_{\mathbb{R}} = \{r \in \mathbb{Q} \mid r < 1\}$. Consider the linear ordering $<$ on $\mathbb{Q}$. The rational numbers 1, 2, 1.5 are upper bounds of B in $\mathbb{Q}$. The set B does not have a lower bound in $\mathbb{Q}$ (as it is closed downwards).

Example 5.16: If B has a greatest element x, then x is an upper bound of B. Similarly, a least element of B is a lower bound of B.

Example 5.17: Every Dedekind cut x is bounded above in $\mathbb{Q}$.



Proof. Let x be a Dedekind cut. Since $x \subsetneq \mathbb{Q}$, there is $q \in \mathbb{Q}$ such that $q \notin x$. Claim: q is an upper bound of x.
Indeed, for any $r \in x$, if $r \nleq q$, i.e., $q < r$, then $q \in x$, as x is closed downwards. A contradiction. □

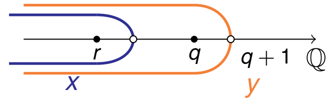
Fact 5.18

Let x be a Dedekind cut, and $q \in \mathbb{Q}$. Then q is an upper bound of x in $\mathbb{Q}$ iff $q \notin x$.

Proof. "$\Longleftarrow$": Follows from the proof of Example 5.17.
"$\Longrightarrow$": Let q be an upper bound of x in $\mathbb{Q}$. If $q \in x$, then q is a greatest element of x, which contradicts x being a Dedekind cut. □

Lemma 5.19

For every $x \in \mathbb{R}$, there exists $y \in \mathbb{R}$ such that $x <_{\mathbb{R}} y$.



Proof. Since x is a Dedekind cut, there exists $q \in \mathbb{Q}$ such that q is an upper bound of x. The set

$$y = (q+1)_{\mathbb{R}} = \{r \in \mathbb{Q} \mid r < q+1\}$$

is a real number.

Claim: $x <_{\mathbb{R}} y$, i.e., $x \subsetneq y$.

Indeed, $r \in x \Longrightarrow r \leq q < q+1 \Longrightarrow r \in y$. Hence $x \subseteq y$.

To see that $x \neq y$, observe that $q \in y$, but $q \notin x$ by Fact 5.18. □

Theorem 5RA

The relation $<_{\mathbb{R}}$ is a linear ordering on $\mathbb{R}$.

Proof. As set inclusion is transitive, the relation $<_{\mathbb{R}}$ is transitive. It remains to show that $<_{\mathbb{R}}$ satisfies trichotomy on $\mathbb{R}$.

For any $x, y \in \mathbb{R}$, obviously *at most* one of the following alternatives

$$x \subset y, \quad x = y, \quad y \subset x$$

holds. We must show that at least one holds.

Suppose $x \nsubseteq y$. We show that $y \subset x$.

By assumption, there is a rational $r \in x \setminus y$. By the Fact 5.18, r is an upper bound of y, thus $q < r$ for all $q \in y$, which gives $q \in x$ as $r \in x$ and x is downwards closed. Hence $y \subset x$. □

Example 5.18: Let $x, y \in \mathbb{R}$.

- $x \cup y$ is a real number.

  Proof. Since $<_{\mathbb{R}}$ on $\mathbb{R}$ is a linear ordering, we have exactly one of the following holds

  $$x \subset y, \quad x = y, \quad y \subset x.$$

  If $x \subset y$, then $x \cup y = y \in \mathbb{R}$. If $x = y$, then $x \cup y = x \in \mathbb{R}$. If $y \subset x$, then $x \cup y = x \in \mathbb{R}$. □

- $x \cap y$ is a real number.

  Proof. Analogously, if $x \subset y$, then $x \cap y = x \in \mathbb{R}$. If $x = y$, then $x \cap y = y \in \mathbb{R}$. If $y \subset x$, then $x \cap y = y \in \mathbb{R}$. □

Definition 5.20

The least upper bound of B, or the supremum of B in a set A (denoted by $\sup B$) is the upper bound of B in A that is less than any other upper bound.
The greatest lower bound of B, or the infimum of B in a set A (denoted by $\inf B$) is the lower bound of B in A that is greater than any other lower bound.

Example 5.19: Let B be a subset of A. If B has a greatest element x, then $\sup B = x$ in A, as x is an upper bound of x and $x < y$ for any other upper bound $y \in A$.

Similarly, if B has a least element y, then $\inf B = y$ in A.

Example 5.20: For an arbitrary $q \in \mathbb{Q}$, consider the Dedekind cut $q_{\mathbb{R}} = \{r \in \mathbb{Q} \mid r < q\}$. Then, in $\mathbb{Q}$ we have $\sup q_{\mathbb{R}} = q$.

Proof. Since $q \notin q_{\mathbb{R}}$, by Fact 5.18, q is an upper bound of $q_{\mathbb{R}}$. For any upper bound p of $q_{\mathbb{R}}$, again by Fact 5.18, we know that $p \notin q_{\mathbb{R}}$, i.e., $p \not< q$, and thus $p \geq q$, as desired. □

Example 5.21: Consider the set $x = \mathbb{Q} \setminus q_{\mathbb{R}}$. Then, in $\mathbb{Q}$ we have $\inf x = q$.

Proof. Clearly $q \in x$ and q is the least element of x, thus $\inf x = q$ follows. □

Example 5.22: Any non-empty finite subset of $\mathbb{R}$ is bounded above and has a least upper bound in $\mathbb{R}$.

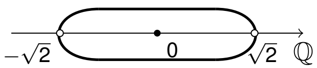
Proof. Let $A = \{x_1, \ldots, x_n\}$ be a finite set of real numbers. Since $<_{\mathbb{R}}$ (or $\subset$) on $\mathbb{R}$ is a linear ordering, we have that

$$x_{i_1} \subset x_{i_2} \subset \cdots \subset x_{i_n},$$

where $i_1, \ldots, i_n \in \{1, \ldots, n\}$. Clearly, x_{i_n} is the greatest element of A. It follows that A is bounded above and $\sup A = x_{i_n} \in \mathbb{R}$. □

Remark: In fact, $\bigcup A = x_1 \cup x_2 \cup \cdots \cup x_n = x_{i_n}$, and $\max A = \sup A = \bigcup A$.

Example 5.23: The set $x = \{r \in \mathbb{Q} \mid r^2 < 2\}$ is a bounded subset of $\mathbb{Q}$, but it has no least upper bound in $\mathbb{Q}$, as $\sup x = \sqrt{2} \notin \mathbb{Q}$.



Theorem 5RB

Any bounded non-empty subset of $\mathbb{R}$ has a least upper bound in $\mathbb{R}$.

Proof. Let A be a bounded non-empty subset of $\mathbb{R}$. We show that $\bigcup A$ is the least upper bound of A in $\mathbb{R}$.

By definition of union, we know that $x \in A$ implies $x \subseteq \bigcup A$, thus $\bigcup A$ is an upper bound of A. For any upper bound z of A, $x \subseteq z$ for all $x \in A$, from which it follows that $\bigcup A \subseteq z$. Hence $\sup A = \bigcup A$.

It remains to show that $\bigcup A \in \mathbb{R}$, i.e., to show that $\bigcup A$ is a Dedekin cut. Since $A \neq \emptyset$ and $x \neq \emptyset$ for all $x \in A$, $\bigcup A \neq \emptyset$. Moreover, $\bigcup A \subsetneq \mathbb{Q}$ since $\bigcup A \subseteq z \subsetneq \mathbb{Q}$, where $z \in \mathbb{R}$ is an upper bound of A.

$\bigcup A$ is closed downwards: For any $q \in \bigcup A$ and $r < q$, there exists $x \in A$ such that $r < q \in x$. Since x is closed downwards, we derive $r \in x$, thereby $r \in \bigcup A$.

$\bigcup A$ has no greatest element: Assume that $\bigcup A$ has a greatest element a. As $a \in \bigcup A$, there exists $x \in A$ such that $a \in x$. For any $b \in x$, we have that $b \in \bigcup A$, thus $b \leq a$. But this means that a is also a greatest element of x, contradicting x being a Dedekind cut. □

Definition 5.21 (Addition)

Define a binary operation $+_{\mathbb{R}}$ on $\mathbb{R}$ as: for any $x, y \in \mathbb{R}$,

$$x +_{\mathbb{R}} y = \{q + r \mid q \in x \wedge r \in y\}$$

Example 5.24: $1_{\mathbb{R}} +_{\mathbb{R}} 2_{\mathbb{R}} = 3_{\mathbb{R}}$.

Proof. Recall that

$$1_{\mathbb{R}} = \{r \in \mathbb{Q} \mid r < 1\} \quad \text{and} \quad 2_{\mathbb{R}} = \{r \in \mathbb{Q} \mid r < 2\}.$$

By definition,

$$\begin{aligned} 1_{\mathbb{R}} +_{\mathbb{R}} 2_{\mathbb{R}} &= \{q + r \mid q \in 1_{\mathbb{R}} \wedge r \in 2_{\mathbb{R}}\} \\ &= \{q + r \mid q < 1 \wedge r < 2 \wedge q, r \in \mathbb{Q}\} \\ &= \{q + r \mid q + r < 3 \wedge q, r \in \mathbb{Q}\} \\ &= \{s \in \mathbb{Q} \mid s < 3\} \\ &= 3_{\mathbb{R}}. \end{aligned}$$

□

Lemma 5RC

For any $x, y \in \mathbb{R}$, we have that $x +_{\mathbb{R}} y \in \mathbb{R}$.

Proof. Since both x and y are non-empty subsets of $\mathbb{Q}$ with no greatest element, $x +_{\mathbb{R}} y$ is also a non-empty subset of $\mathbb{Q}$ with no greatest element.

To show that $x +_{\mathbb{R}} y \neq \mathbb{Q}$, pick some $q' \in \mathbb{Q} \setminus x$ and $r' \in \mathbb{Q} \setminus y$. Then

$$\begin{aligned} q \in x \text{ and } r \in y &\implies q < q' \text{ and } r < r' \qquad \text{(by Fact 5.18)} \\ &\implies q + r < q' + r'. \end{aligned}$$

It follows that every member $q + r$ of $x +_{\mathbb{R}} y$ is strictly less than $q' + r'$. Hence $q' + r' \notin x +_{\mathbb{R}} y$.

To show that $x +_{\mathbb{R}} y$ is closed downwards, consider any $p, q, r \in \mathbb{Q}$ such that $q \in x$, $r \in y$ and

$$p < q + r \in x +_{\mathbb{R}} y. \qquad (*)$$

We want to show that $q + (p - q) = p \in x +_{\mathbb{R}} y$. It follows from $(*)$ that $p - q < r \in y$, which by downwards closure of y implies that $p - q \in y$. Hence $p = q + (p - q) \in x +_{\mathbb{R}} y$. □

Theorem 5RD

Let $x, y, z \in \mathbb{R}$.

Commutative Law $x +_{\mathbb{R}} y = y +_{\mathbb{R}} x$

Associative Law $(x +_{\mathbb{R}} y) +_{\mathbb{R}} z = x +_{\mathbb{R}} (y +_{\mathbb{R}} z)$

Proof. We have that

$$\begin{aligned} x +_{\mathbb{R}} y &= \{r + s \mid r \in x \wedge s \in y\} \\ &= \{s + r \mid r \in x \wedge s \in y\} \qquad (\because + \text{ on } \mathbb{Q} \text{ is commutative}) \\ &= y +_{\mathbb{R}} x, \end{aligned}$$

and

$$\begin{aligned} (x +_{\mathbb{R}} y) +_{\mathbb{R}} z &= \{p + t \mid p \in x + y \wedge t \in z\} \\ &= \{(r + s) + t \mid r \in x \wedge s \in y \wedge t \in z\} \\ &= \{r + (s + t) \mid r \in x \wedge s \in y \wedge t \in z\} \\ &\qquad (\because + \text{ on } \mathbb{Q} \text{ is associative}) \\ &= \{r + q \mid r \in x \wedge q \in y + z\} \\ &= x +_{\mathbb{R}} (y +_{\mathbb{R}} z). \end{aligned}$$

□

Theorem 5RE

$x +_{\mathbb{R}} 0_{\mathbb{R}} = x$ for any $x \in \mathbb{R}$.

Proof. Putting $y = \{r + s \mid r \in x \wedge s < 0\} = x +_{\mathbb{R}} 0_{\mathbb{R}}$, it suffices to show that $y = x$.

For any $r + s \in y$ with $r \in x$ and $s < 0$, since $r + s < r$, we obtain $r + s \in x$ by downwards closure. Hence $y \subseteq x$.

Conversely, for any $p \in x$, since x does not have a greatest element, there exists r with $p < r \in x$. It follows that

$$p = r + (p - r) \in y.$$

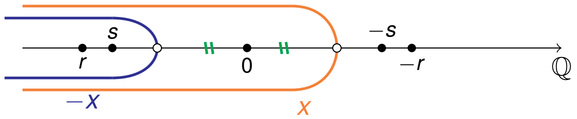
as $p - r < 0$. Hence $x \subseteq y$. □

Definition 5.22

For any $x \in \mathbb{R}$, define the additive inverse of x as

$$-x := \{r \in \mathbb{Q} \mid \exists s \in \mathbb{Q}(s > r \wedge -s \notin x)\}.$$

The idea:



We want $\forall r \in \mathbb{Q}(r \in -x \implies -r \notin x)$.

A first attempt: $-x := \{r \in \mathbb{Q} \mid -r \notin x\}$.

But, if, say, $x = 2_{\mathbb{R}}$, then

$$-(2_{\mathbb{R}}) = \{r \in \mathbb{Q} \mid -r \notin 2_{\mathbb{R}}\} = \{r \in \mathbb{Q} \mid -r \geq 2\} = \{r \in \mathbb{Q} \mid r \leq -2\},$$

which contains a greatest element, thus is not a Dedekind cut.

A modified definition:

$$\begin{aligned} -(2_{\mathbb{R}}) &:= \{r \in \mathbb{Q} \mid \exists s \in \mathbb{Q}(s > r \wedge -s \notin 2_{\mathbb{R}})\} \\ &= \{r \in \mathbb{Q} \mid \exists s \in \mathbb{Q}(s > r \wedge s \leq -2)\} \\ &= \{r \in \mathbb{Q} \mid r < -2\} = (-2)_{\mathbb{R}} \end{aligned}$$

Example 5.25: $-(q_{\mathbb{R}}) = (-q)_{\mathbb{R}}$ for any $q \in \mathbb{Q}$.

Proof. Recall that

$$q_{\mathbb{R}} = \{r \in \mathbb{Q} \mid r < q\} \quad \text{and} \quad (-q)_{\mathbb{R}} = \{r \in \mathbb{Q} \mid r < -q\}.$$

By definition,

$$\begin{aligned} -(q_{\mathbb{R}}) &= \{r \in \mathbb{Q} \mid \exists s \in \mathbb{Q}(s > r \wedge -s \notin q_{\mathbb{R}})\} \\ &= \{r \in \mathbb{Q} \mid \exists s \in \mathbb{Q}(s > r \wedge -s \geq q)\} \\ &= \{r \in \mathbb{Q} \mid \exists s \in \mathbb{Q}(s > r \wedge s \leq -q)\} \\ &= \{r \in \mathbb{Q} \mid \exists s \in \mathbb{Q}(r < s \leq -q)\} \\ &= \{r \in \mathbb{Q} \mid r < -q\} \\ &= (-q)_{\mathbb{R}}. \end{aligned}$$

□

As a consequence, $-(-(q_{\mathbb{R}})) = -(-q)_{\mathbb{R}} = (--q)_{\mathbb{R}} = q_{\mathbb{R}}$ for all $q \in \mathbb{Q}$.

Theorem 5RF

$-x \in \mathbb{R}$ for any $x \in \mathbb{R}$.

Proof. We first show that $\emptyset \neq -x \neq \mathbb{Q}$.

Since $x \neq \mathbb{Q}$, $\exists t \in \mathbb{Q} \setminus x$. For any rational $r < -t$, since $-(-t) = t \notin x$, we have $r \in -x$. Hence $-x \neq \emptyset$.

To show that $-x \neq \mathbb{Q}$, take any $p \in x$. We claim that $-p \notin -x$. Indeed, for any $s \in \mathbb{Q}$, $\quad s > -p \implies -s < p \in x \implies -s \in x \implies -p \notin -x$.

Secondly, $-x$ is closed downward. Indeed, for any $q < r \in -x$, we have $\exists s > r(-s \notin x)$. Hence $\exists s > q(-s \notin x)$, thereby $q \in -x$.

Finally, $-x$ has no largest element. Indeed,

$$\begin{aligned} r \in -x &\implies \exists s > r(-s \notin x) \\ &\implies \exists s > p > r(-s \notin x) \text{ for some } p \in \mathbb{Q} \quad (\text{since } \mathbb{Q} \text{ is dense}) \\ &\implies r < p \in -x. \end{aligned}$$

This completes the proof. □

It is proved in Theorem 5RF of the book that $x + (-x) = 0_{\mathbb{R}}$ for every $x \in \mathbb{R}$, but we omit the proof here.

The absolute value $|x|$ of a real number x is defined to be $\max\{x, -x\}$. Equivalently:

Definition 5.23 (absolute value)

For any $x \in \mathbb{R}$, define

$$|x| := x \cup -x.$$

Example 5.26: $|(-1)_{\mathbb{R}}| = (-1)_{\mathbb{R}} \cup -(-1)_{\mathbb{R}} = (-1)_{\mathbb{R}} \cup 1_{\mathbb{R}} = 1_{\mathbb{R}}$.

Example 5.27: $|0_{\mathbb{R}}| = 0_{\mathbb{R}} \cup -(0_{\mathbb{R}}) = 0_{\mathbb{R}} \cup (-0)_{\mathbb{R}} = 0_{\mathbb{R}} \cup 0_{\mathbb{R}} = 0_{\mathbb{R}}$.

Lemma 5.24

For any $x \in \mathbb{R}$, we have $|x| \in \mathbb{R}$.

Proof. Since $x, -x \in \mathbb{R}$, we know by Example 5.22 that $|x| = x \cup -x \in \mathbb{R}$. □

Exercise: $0_{\mathbb{R}} \leq_{\mathbb{R}} |x|$ for all $x \in \mathbb{R}$.