Elements of Set Theory

Chapter 3: Relations and Functions

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Teacher: Fan Yang

Theorem 3P

Let $\equiv$ be an equivalence relation on a nonempty set A. The quotient set

$$A/_{\equiv} = \{[x] : x \in A\}$$

is a partition of A.

Proof.

1. For any $x, y \in A$, by Proposition 3.13 (iii),

$$[x] \neq [y] \Longrightarrow [x] \cap [y] = \emptyset.$$

2. $A = \bigcup_{x\in A}[x]$. Indeed, $[x] = \{a \in A : x \equiv a\} \subseteq A$ for any $x \in A$, which leads to $\bigcup_{x\in A}[x] \subseteq A$.
   On the other hand, $A \subseteq \bigcup_{x\in A}[x]$ since

$$y \in A \Longrightarrow y \in [y] \subseteq \bigcup_{x\in A}[x] \Longrightarrow y \in \bigcup_{x\in A}[x].$$

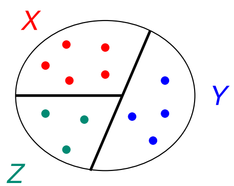
□

The converse of the preceding theorem also holds:

Theorem 3.16

Let Π be a partition of a nonempty set A. The relation $\equiv$ defined as follows is an equivalence relation on A: for any $x, y \in A$,

$$x \equiv y \iff \text{there exists } X \in \Pi \text{ such that } x, y \in X.$$



Proof. By definition, $\equiv$ is clearly reflexive and symmetric. It remains to show that $\equiv$ is transitive.
For any $x, y, z \in A$ such that $x \equiv y$ and $y \equiv z$, by the definition of $\equiv$, there exist $X, Y \in \Pi$ such that

$$x, y \in X \text{ and } y, z \in Y.$$

Hence $y \in X \cap Y \neq \emptyset$, which implies $X = Y$ as Π is a partition. It then follows that $x, z \in X = Y$, thereby $x \equiv z$. □

Functions

The squaring function $f : \mathbb{R} \to \mathbb{R}$ is defined as $f(x) = x^2$. The action of f on particular elements of $\mathbb{R}$ can be described by writing

$$-2 \mapsto 4, \quad \sqrt{2} \mapsto 2, \quad 2 \mapsto 4, \quad 3 \mapsto 9, \text{ etc.}$$

Each individual action can be represented by an ordered pair:

$$(-2, 4), \quad (\sqrt{2}, 2), \quad (2, 4), \quad (3, 9), \text{ etc.}$$

The set

$$F = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid y = x^2\}$$

of all such ordered pairs adequately represents the squaring function f, and is called the *graph* of f. In set theory, we simply take this set F of ordered pairs to *be* the function f.

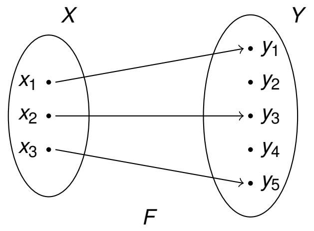
Clearly, the set $F \subseteq \mathbb{R} \times \mathbb{R}$ is a relation such that for any $x \in$dom F, there is a unique y such that $(x, y) \in F$.

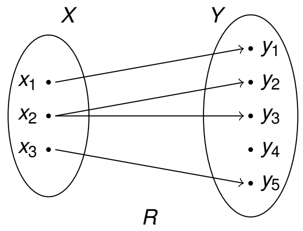
Definition 3.17

A function is a binary relation F such that for each $x \in$dom F, there is a unique y such that $(x, y) \in F$.

The unique y is called the value of F at x, denoted by $F(x)$.

If $\operatorname{dom} F = X$ and $\operatorname{ran} F \subseteq Y$, then we write $F : X \to Y$ and say that F is a function from X into Y, or that F maps X into Y.





For example:

- The relation $F = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid y = x^2\}$ is a function, $\operatorname{dom} F = \mathbb{R}$ and $\operatorname{ran} F = \mathbb{R}^+ \cup \{0\}$. We write $F : \mathbb{R} \to \mathbb{R}$.
- The relation $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x^2 + y^2 = 4\}$ is not a function, since e.g., $(0, 2), (0, -2) \in R$.

The domain of a function can be a set of ordered pairs or n-tuples. For example, addition is a function $+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. In this case, in place of $+((a, b))$, we write either $+(a, b)$ or $a + b$.

Fact 3.18

If F and G are functions for which $\operatorname{dom} F = \operatorname{dom} G$ and

$$F(x) = G(x)$$

for every $x \in \operatorname{dom} F$, then $F = G$.

Proof. Indeed, by the definition of functions, we have that

$$\begin{aligned} (x, y) \in F &\iff x \in \operatorname{dom} F \text{ and } y = F(x) \\ &\iff x \in \operatorname{dom} G \text{ and } y = G(x) \quad \text{(by assumption)} \\ &\iff (x, y) \in G. \end{aligned}$$

Hence $F = G$. □

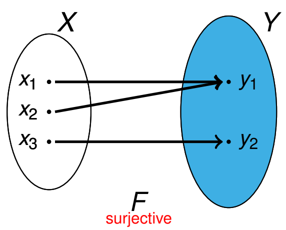
Definition 3.19

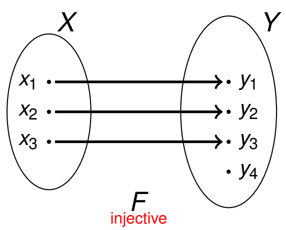
Let $F : X \to Y$ be a function from X into Y.

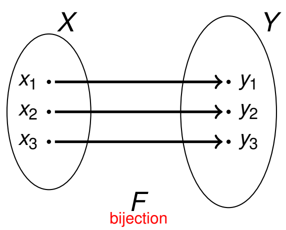
- *If $\operatorname{ran} F = Y$, then we say that F is surjective or F is a function from X onto Y.*
- *If for any $x_1, x_2 \in X$,*

  $$x_1 \neq x_2 \implies F(x_1) \neq F(x_2),$$

  then we say that F is one-to-one or injective.
- *If F is both surjective and injective, then F is called a bijection.*







Note: A function $F : X \to Y$ is surjective iff for each $y \in Y$, there exists $x \in X$ such that $F(x) = y$.

For example:

Consider the function $F : \mathbb{R} \to \mathbb{R}$, defined as $F(x) = x^2$.

- F is not surjective, since $\operatorname{ran} F = \mathbb{R}^+ \cup \{0\} \neq \mathbb{R}$.
- F is not injective, since e.g. $F(2) = 4 = F(-2)$.

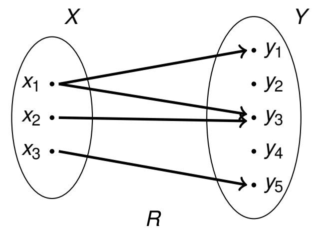
Consider the function $G : \mathbb{R} \to \mathbb{R}$, defined as $G(x) = x^3$.

- G is surjective, since for each $y \in \mathbb{R}$, we have that $G(\sqrt[3]{y}) = y$.
- G is injective, since $G(x_1) = G(x_2) \implies x_1^3 = x_2^3 \implies x_1 = x_2$.

Thus G is a bijection.

Definition 3.20

A relation R is said to be single-rooted iff for each $y \in \operatorname{ran} R$, there is only one x such that xRy.



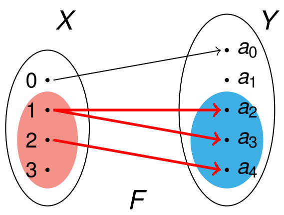
not single-rooted

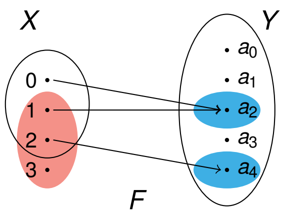
Clearly, if F is a function, then F is single-rooted iff F is one-to-one.

Definition 3.21

Let F be an arbitrary relation, and A a set.

- *The restriction of F to A is the set*
$$F \upharpoonright A = \{(x, y) \mid xFy \wedge x \in A\}.$$
- *The image of A under F is the set*
$$F[A] = ran\,(F \upharpoonright A) = \{y \mid \exists x \in A(xFy)\}.$$
In particular, if F is a function, then $F[A] = \{F(x) \mid x \in A\}$.





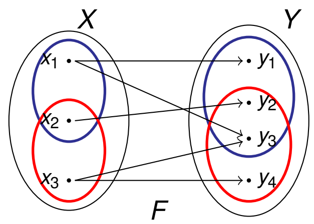
Theorem 3K

Let F be a relation.

(a) $F[A \cup B] = F[A] \cup F[B]$.

(b) $F[A \cap B] \subseteq F[A] \cap F[B]$. Equality holds if F is single-rooted.

(b) $F[A] - F[B] \subseteq F[A - B]$. Equality holds if F is single-rooted.



Proof. (a).
$$\begin{aligned} y \in F[A \cup B] &\iff \exists x \in A \cup B \text{ s.t. } xFy \\ &\iff (\exists x \in A \text{ s.t. } xFy\ ) \text{ or } (\exists x \in B \text{ s.t. } xFy) \\ &\iff y \in F[A] \text{ or } y \in F[B] \\ &\iff y \in F[A] \cup F[B]. \end{aligned}$$

(b). $y \in F[A \cap B] \iff \exists x \in A \cap B$ s.t. xFy

$$\begin{aligned} &\implies (\exists x \in A \text{ s.t. } xFy) \text{ and } (\exists x' \in B \text{ s.t. } x'Fy) \\ &\iff y \in F[A] \text{ and } y \in F[B] \\ &\iff y \in F[A] \cap F[B]. \end{aligned}$$

If F is single-rooted, then in the second line of the above expression, $x = x'$, which means that the second arrow is reversible.

(c). $y \in F[A] - F[B] \iff y \in F[A]$ and $y \notin F[B]$

$$\begin{aligned} &\iff (\exists x \in A \text{ s.t. } xFy) \text{ and } (\neg \exists x \in B(xFy)) \\ &\implies \exists x \in A - B \text{ s.t. } xFy \\ &\iff y \in F[A - B]. \end{aligned}$$

If F is single-rooted, then the thrid arrow is reversible. □

Definition 3.22

Let F be a relation. The inverse of F is the set

$$F^{-1} = \{(y, x) \mid xFy\}.$$

Note:

- $(F^{-1})^{-1} = F$
- In general, even if F is a function, F^{-1} is not necessarily a function, but F^{-1} is a single-rooted relation, i.e., for any $x \in \operatorname{ran} F^{-1}$, there is only one y such that $yF^{-1}x$.

Since the inverse of a function is always single-rooted, the following corollary is an immediate consequence of the Theorem 3K.

Corollary 3L

For any function G and sets A, B:

(a) $G^{-1}[A \cup B] = G^{-1}[A] \cup G^{-1}[B]$.

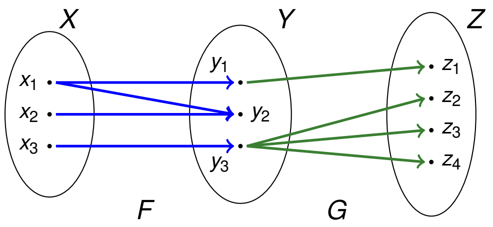
(b) $G^{-1}[A \cap B] = G^{-1}[A] \cap G^{-1}[B]$.

(b) $G^{-1}[A] - G^{-1}[B] = G^{-1}[A - B]$

Definition 3.23

Let F, G be arbitrary relations. The composition of F and G is the set

$$G \circ F = \{(x, z) \mid \exists y(xFy \wedge yGz)\}.$$



Theorem 3H

If $F : X \to Y$ and $G : Y \to Z$ are functions, then $G \circ F : X \to Z$ is a function, and $G \circ F(x) = G(F(x))$ for all $x \in X$.

Proof. We first check that $G \circ F$ is a function. Suppose that $x(G \circ F)z$ and $x(G \circ F)z'$. Then there exist $y, y' \in Y$ such that

$$xFy \wedge yGz \text{ and } xFy' \wedge y'Gz'.$$

Since F is a function, we derive that $y = y'$. Furthermore, since G is a function, yGz and yGz' implies that $z = z'$, as desired.

For any $x \in X$, we have that $(x, F(x)) \in F$ and $(F(x), G(F(x))) \in G$, thus $(x, G(F(x))) \in G \circ F$, which implies that $G \circ F(x) = G(F(x))$, as $G \circ F$ is a function. □

Theorem 3I

For any relations F and G,

$$(G \circ F)^{-1} = F^{-1} \circ G^{-1}.$$

Proof. For any ordered pair (z, x),

$$\begin{aligned} (z, x) \in (G \circ F)^{-1} &\iff (x, z) \in G \circ F \\ &\iff xFy \wedge yGz, \quad \text{for some } y \\ &\iff yF^{-1}x \wedge zG^{-1}y, \quad \text{for some } y \\ &\iff (z, x) \in F^{-1} \circ G^{-1} \end{aligned}$$

□

Definition 3.24

Given a nonempty set A. The **identity function** $\mathrm{id}_A : A \to A$ *on A is defined as*

$$\mathrm{id}_A = \{(x, x) \mid x \in A\}.$$

Clearly, $\mathrm{id}_A(x) = x$ for all $x \in A$.

Theorem 3J

Let $F : X \to Y$ be a function, $A \neq \emptyset$.

(a) *There exists a function $G : Y \to X$ (a "left inverse") such that $G \circ F = \mathrm{id}_X$ iff F is one-to-one.*

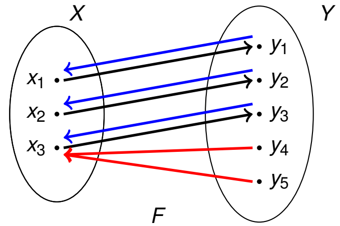
(b) *There exists a function $H : Y \to X$ (a "right inverse") such that $F \circ H = \mathrm{id}_Y$ iff F is surjective.*

Proof. (a) "$\Longrightarrow$": Suppose G is a left inverse of F. For any $a, b \in X$,

$$F(a) = F(b) \Longrightarrow G \circ F(a) = G \circ F(b) \Longrightarrow \mathrm{id}_X(a) = \mathrm{id}_X(b) \Longrightarrow a = b.$$

"$\Longleftarrow$": Suppose F is one-to-one. It is easy to see that F^{-1} is a function from $\mathrm{ran}\, F \subseteq Y$ into X. We extend F^{-1} to a function $G : Y \to X$ with the required property. Pick an arbitrary $a \in X$. Define

$$G(y) = \begin{cases} F^{-1}(y), & \text{if } y \in \mathrm{ran\,F} \\ a, & \text{otherwise.} \end{cases}$$



Now, for any $x \in X$, $F(x) \in \mathrm{ran}\, F$, thereby

$$G \circ F(x) = G(F(x)) = F^{-1}(F(x)) = x = \mathrm{id}_X(x).$$

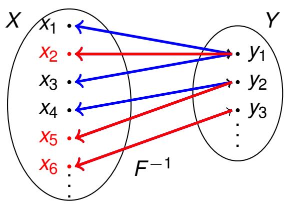
Hence $G \circ F = \mathrm{id}_X$, namely, G is the left inverse of F.

(b) "$\Longrightarrow$": Suppose H is a right inverse of F. For any $y \in Y$,

$$y = \mathrm{id}_Y(y) = F \circ H(y) = F(H(y)),$$

where $H(y) \in X$, thus F is surjective.

"$\Longleftarrow$": Suppose F is surjective. The idea is that for each $y \in Y = \mathrm{ran}\, F$, we choose an element $x \in X$ such that $F(x) = y$, and let $H(y) = x$. For *one* y, we know there exists an appropriate x, but *in general* we have no way of defining any particular choice of x. So to guarantee that we indeed can form the function $H : Y \to X$, an axiom is needed:



Axiom of Choice (First form)

For any relation R, there is a function $H \subseteq R$ with $\mathrm{dom}\, H = \mathrm{dom}\, R$.

Now, by Axiom of Choice, for the relation F^{-1}, there is a function $H \subseteq F^{-1}$ such that dom H =dom $F^{-1} = Y$.

Next, we check that H is a right inverse of F. For any $y \in Y$,

$$(y, H(y)) \in H \subseteq F^{-1} \Longrightarrow (y, H(y)) \in F^{-1} \Longrightarrow (H(y), y) \in F,$$

thus $F(H(y)) = y = \mathrm{id}_Y(y)$. □

Definition 3.25

Let A and B be sets. The set of all functions F from A into B is denoted by ${}^{A}B$. That is,

$$ {}^{A}B = \{F \subseteq A \times B \mid F : A \to B \text{ is a function}\}. $$

Example 3.1: Let $\omega = \{0, 1, 2, \ldots\}$. Then ${}^{\omega}\{0, 1\}$ is the set of all function $f : \omega \to \{0, 1\}$. Such an f can be viewed as an infinite binary sequence $\langle f(0), f(1), f(2), \ldots\rangle$ of 0's and 1's.

Example 3.2: We have that

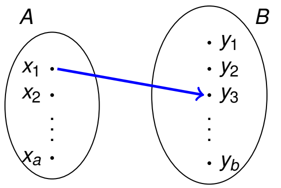
$$ {}^{A}\emptyset = \{f \subseteq A \times \emptyset \mid f : A \to \emptyset\} = \emptyset $$

for any set $A \neq \emptyset$. On the other hand,

$$ {}^{\emptyset}B = \{f \subseteq \emptyset \times B \mid f : \emptyset \to B\} = \{\emptyset\} $$

for any set B. In particular, ${}^{\emptyset}\emptyset = \{\emptyset\}$.

Example 3.3: If A and B are finite sets with a and b elements, respectively, then the set ${}^{A}B$ has b^a elements.



Note that for each of the a elements of A, we can choose among b points in B into which it could be mapped. Thus, the number of ways of making all a such choices is $\underbrace{b \cdot \cdots \cdot b}_{a \text{ times}}$.

Infinite Cartesian Products

For a function $F : I \to A$, we sometimes write F_i for the value $F(i)$.

We sometimes represent a set $\mathscr{F}$ of sets as a collection of indexed sets:

$$ \mathscr{F} = \{F_i \mid i \in I\}. $$

This set $\mathscr{F}$ corresponds to a function F such that $I \subseteq \operatorname{dom} F$ and

$$ F(i) = F_i, \text{ for each } i \in I. $$

We have defined finite Cartesian products as

$$A_1 \underbrace{\times \cdots \times}_{n} A_n = (A_1 \underbrace{\times \cdots \times}_{n-1} A_{n-1}) \times A_n$$

for each n. In general, we define infinite Cartesian product as follows:

Definition 3.26

Let I be a set, and H a function such that $I \subseteq \operatorname{dom} H$. Define the Cartesian product of the H_i's for all $i \in I$ as

$$\prod_{i \in I} H_i = \{f : I \to \bigcup_{i \in I} H_i \mid \forall i \in I(f(i) \in H_i)\}.$$

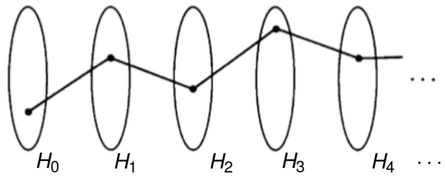
Example 3.4: If for every $i \in I$, we have $H_i = A$ for some fixed set A, then

$$\prod_{i \in I} H_i = \prod_{i \in I} A = \{f : I \to \bigcup_{i \in I} A \mid \forall i \in I(f(i) \in A)\}$$
$$= \{f : I \to A \mid \forall i \in I(f(i) \in A)\} = {}^I A.$$

Example 3.5: If the index set $I = \omega = \{0, 1, 2, \ldots\}$, then

$$\prod_{n \in \omega} H_n = \{f : \omega \to \bigcup_{n \in \omega} H_n \mid \forall n \in \omega(f(n) \in H_n)\}.$$

So $\prod_{n \in \omega} H_n$ consists of "ω-sequences" that have for their nth term some member of H_n. A typical member is a "thread" in the following picture that selects a point from each set.



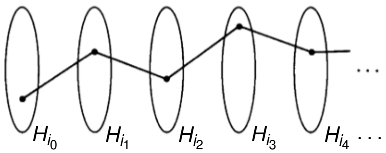
Example 3.6: In particular, if $I = \emptyset$, then

$$\prod_{i \in \emptyset} H_i = \{f : \emptyset \to \bigcup_{i \in \emptyset} H_i \mid \forall i \in \emptyset(f(i) \in H_i)\} = \{f : \emptyset \to \emptyset\} = \{\emptyset\}.$$

If H_i is empty for some $i \in I$, then $\prod_{i \in I} H_i = \emptyset$. Conversely, if $H_i \neq \emptyset$ for each $i \in I$, do we know that $\prod_{i \in I} H_i \neq \emptyset$? This requires an axiom:

Axiom of Choice (2nd form)

For any set I and any function H with $I \subseteq \operatorname{dom} H$, if $H_i \neq \emptyset$ for all $i \in I$, then $\prod_{i \in I} H_i \neq \emptyset$.

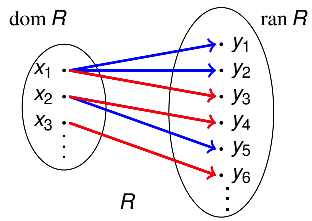


We have so far encountered two forms of Axiom of Choice. The two forms are equivalent.

Recall the first form:

Axiom of Choice (First form)

For any relation R, there is a function $F \subseteq R$ with $\operatorname{dom} F = \operatorname{dom} R$.



Theorem 6M

The First and the Second form of Axiom of Choice are equivalent, that is, the following two statements are equivalent:

(1) For any relation R, there is a function $F \subseteq R$ with $\operatorname{dom} F = \operatorname{dom} R$.

(2) For any set I and any function H with $I \subseteq \operatorname{dom} H$, if $H_i \neq \emptyset$ for all $i \in I$, then $\prod_{i \in I} H_i \neq \emptyset$.

Recall:

- A function $F : I \to A$ is a set
$$F = \{(i, F(i)) \mid i \in I\}.$$
- Let R be a relation. For any set A,
$$R[A] = \{y \mid \exists x \in A((x, y) \in R)\}.$$
In particular, for every $i \in \operatorname{dom} R$,
$$R[\{i\}] = \{y \mid (i, y) \in R\},$$
i.e., $y \in R[\{i\}] \iff (i, y) \in R$.

Proof. (of Theorem 6M) (1)$\Longrightarrow$(2): Suppose I is a set and H is a function such that $I \subseteq \operatorname{dom} H$ and $H_i \neq \emptyset$ for all $i \in I$. We want to show that
$$\prod_{i \in I} H_i = \{f : I \to \bigcup_{i \in I} H_i \mid \forall i \in I(f(i) \in H_i)\} \neq \emptyset.$$

If $I = \emptyset$ then $\prod_{i \in I} H_i = \{\emptyset\} \neq \emptyset$. Now, assume $I \neq \emptyset$. In order to utilize (1), consider the relation
$$R = \{(i, x) \mid i \in I \wedge x \in H_i\}.$$

By (1), there is a function $F \subseteq R$ with $\operatorname{dom} F = \operatorname{dom} R = I$. For each $i \in I$, we have that
$$(i, F(i)) \in F \subseteq R \Longrightarrow (i, F(i)) \in R \Longrightarrow F(i) \in H_i.$$

Hence we conclude that $F \in \prod_{i \in I} H_i$, as desired.

(2)$\Longrightarrow$(1): Let R be an arbitrary relation. If $R = \emptyset$, then the empty function $F = \emptyset$ satisfies the condition in (1).

Now, assume $R \neq \emptyset$. Let $\operatorname{dom} R = I$. In order to utilize (2), consider the function $H : I \to \{R[\{i\}] \mid i \in I\}$ defined as
$$H_i = R[\{i\}], \quad \text{for all } i \in I.$$

Since $i \in \operatorname{dom} R$, we have that $R[\{i\}] \neq \emptyset$ for each $i \in I$. By (2), $\prod_{i \in I} H_i \neq \emptyset$, meaning that there exists a function
$$F \in \prod_{i \in I} H_i = \{f : I \to \bigcup_{i \in I} H_i \mid \forall i \in I(f(i) \in H_i)\}. \qquad (*)$$

Clearly, $\operatorname{dom} F = I = \operatorname{dom} R$. It remains to check that $F \subseteq R$.

We have that $(*)$ implies that $F(i) \in H_i = R[\{i\}]$ for any $i \in I$, namely, $(i, F(i)) \in R$ for any $i \in I$, which implies $F \subseteq R$. $\square$

Operations

Definition 3.27

An *n-ary operation* *on a set* A *is a function from* $\underbrace{A \times \cdots \times A}_{n}$ *into* A.

For example

- Addition $+ : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$, multiplication $\cdot : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ are binary operations on $\mathbb{Z}$. Instead of writing $+(m, n)$ and $\cdot(m, n)$, we will write $m + n$ and $m \cdot n$.
- The function $s : \omega \to \omega$ defined by

$$s(n) = n + 1$$

is a unary operation on ω.

Let $\equiv$ be an equivalence relation on $A = \{0, 1, 2\}$ defined by $x \equiv y \iff 2 \mid x - y$. Thus there are two distinct equivalence classes:

$$[0] = [2] = \{0, 2\} \text{ and } [1] = \{1\},$$

and $S/_{\equiv} = \{[0], [1]\} = \{\{0, 2\}, \{1\}\}$.

Consider the following operation $*$ on A:

$$x * y = \min\{x, y\} \text{ for all } x, y \in A.$$

Can the following expression define an operation $\odot$ on the set $A/_{\equiv}$

$$[x] \odot [y] = [x * y] \text{ for all } [x], [y] \in A/_{\equiv}?$$

Answer: No. Since $[0] = [2]$ requires $[0] \odot [1] = [2] \odot [1]$, whereas

$$[0] \odot [1] = [0 * 1] = [0],$$
$$\nparallel$$
$$[2] \odot [1] = [2 * 1] = [1].$$

Let $*$ be an operation on a non-empty set A, and $\equiv$ an equivalence relation on A.

Define an operation $\odot$ on the quotient set $A/_{\equiv}$ by putting

$$[x] \odot [y] = [x * y] \text{ for all } [x], [y] \in A/_{\equiv}.$$

We say that $\odot$ is *well-defined* if for all $x, x', y, y' \in A$,

$$[x] = [x'] \text{ and } [y] = [y'],$$

imply that

$$[x] \odot [y] = [x'] \odot [y'],$$

or equivalently

$$[x * y] = [x' * y'].$$

Example 3.7: Let $+$ be the ordinary addition on $\mathbb{Z}$, and $\equiv$ the equivalence relation on $\mathbb{Z}$ defined by

$$x \equiv y \iff 3 \mid x - y.$$

Define an operation $\oplus$ on the quotient set $\mathbb{Z}/_{\equiv}$ by

$$[x] \oplus [y] = [x + y] \text{ for all } x, y \in \mathbb{Z}.$$

Show that $\oplus$ is well-defined.

Proof. Let $x, x', y, y' \in \mathbb{Z}$ be such that $[x] = [x']$ and $[y] = [y']$. We proceed to show that

$$[x + y] = [x' + y'].$$

From the assumption $[x] = [x']$ and $[y] = [y']$ it follows that

$$3 \mid x - x' \text{ and } 3 \mid y - y',$$

which imply that

$$3 \mid (x - x') + (y - y'), \text{ i.e., } 3 \mid (x + y) - (x' + y'),$$

namely $[x + y] = [x' + y']$. □