Chapter 3: Relations and Functions

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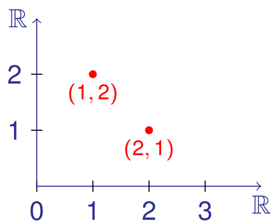
Ordered Pairs

The pair set $\{x, y\}$ is viewed as an unordered pair, since by Extensionality Axiom, $\{x, y\} = \{y, x\}$.

But in many cases, we do want to consider ordered pairs. For example, in the Cartesian coordinate system depicted below, the two points $(1, 2)$ and $(2, 1)$ are different.



$(1, 2) \neq (2, 1)$

In general, we want to define a set denoted by (x, y) that uniquely encodes both what x and y are, and what order they are in. In other words, we require that the set (x, y) can be decomposed uniquely:

$$(x, y) = (u, v) \iff x = u \text{ and } y = v. \qquad (*)$$

In fact, any way of defining (x, y) that satisfies $(*)$ will suffice.

Let us try to find a definition for (x, y).

First attempt: If we define $(x, y) = \{x, y\}$, then for $a \neq b$,

$$(a, b) = \{a, b\} = \{b, a\} = (b, a),$$

which violates $(*)$.

Second attempt: If we define $(x, y) = \{\{x\}, y\}$, then

$$(\{\emptyset\}, \{\emptyset\}) = \{\, \{\{\emptyset\}\},\ \{\emptyset\}\,\} = \{\, \{\emptyset\},\ \{\{\emptyset\}\}\,\} = (\emptyset, \{\{\emptyset\}\}),$$

but $\emptyset \neq \{\emptyset\}$, which also violates $(*)$.

The following is one of the correct definitions:

Definition 3.1 (Kuratowski)

The set (x, y) is defined to be $\{\{x\}, \{x, y\}\}$.

Theorem 3A

$(x, y) = (u, v) \iff x = u$ and $y = v$.

Proof. "$\Longleftarrow$": Clearly, if $x = u$ and $y = v$, then $(x, y) = (u, v)$.
"$\Longrightarrow$": Suppose $(x, y) = (u, v)$, i.e.,

$$\{\{x\}, \{x, y\}\} = \{\{u\}, \{u, v\}\}. \tag{1}$$

Case 1: $x = y$. Then from (1), we know that $\{\{x\}\} = \{\{u\}, \{u, v\}\}$, thus $u = v = x = y$.
Case 2: $x \neq y$. From (1), it follows that

$$\{u\} \in \{\{x\}, \{x, y\}\} \overset{x \neq y}{\Longrightarrow} \{u\} = \{x\} \Longrightarrow u = x.$$

By (1) again, we know that

$$\{x, y\} \in \{\{u\}, \{u, v\}\} \overset{x \neq y}{\Longrightarrow} \{x, y\} = \{u, v\} \overset{x = u}{\Longrightarrow} y = v.$$

□

Lemma 3B

If $x, y \in A$, then $(x, y) \in \wp\wp(A)$.

Proof. Since $x, y \in A$,

$$\{x\}, \{x, y\} \subseteq A, \text{ i.e., } \{x\}, \{x, y\} \in \wp(A),$$

thus

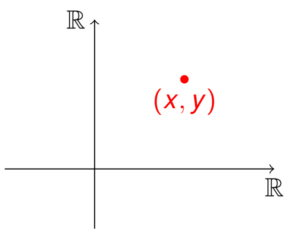
$$\{\{x\}, \{x, y\}\} \subseteq \wp(A), \text{ i.e., } (x, y) \in \wp\wp(A).$$

□

Note: In particular, $(x, x) := \{\{x\}, \{x, x\}\} = \{\{x\}\}$.

We define the *first coordinate* of the ordered pair (x, y) to be x, and the *second coordinate* to be y.

An ordered pair (x, y), where $x, y \in \mathbb{R}$, can be visualized as a point in the real plane.



Corollary 3C

Let A, B be two sets.

$$A \times B = \{(x, y) : x \in A \text{ and } y \in B\}$$

is a set, called the Cartesian product of A and B.

Proof. We have that

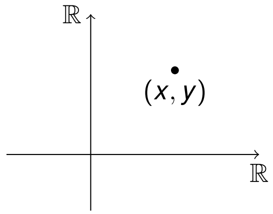
$$\begin{aligned} A \times B &= \{(x, y) \mid x \in A \text{ and } y \in B\} \\ &= \{w \in \wp\wp(A \cup B) \mid \exists x \exists y (w = (x, y) \wedge x \in A \wedge y \in B)\}, \end{aligned}$$

thus by Separation Axiom, $A \times B$ is a set.

□

For example:

- $\mathbb{R} \times \mathbb{R}$ is the real plane:



- if $A = \{a, b, c\}$ and $B = \{x, y\}$, then

$$A \times B = \{(a, x), (a, y), (b, x), (b, y), (c, x), (c, y)\}.$$

- If $A = \emptyset$ or $B = \emptyset$, then $A \times B = \emptyset$.

Remark

We chose to use Kuratowski's definition of ordered pairs (1921):

$$(x, y) := \{\{x\}, \{x, y\}\}.$$

In fact, any definition that satisfies the following condition suffices:

$$(x, y) = (u, v) \iff x = u \text{ and } y = v.$$

Two alternative definitions:

Wiener's definition (1914):

$$(x, y) := \{\,\{\{x\}, \emptyset\}, \{\{y\}\}\,\}.$$

Hausdorff's definition (1914):

$$(x, y) := \{\{x, 1\}, \{y, 2\}\},$$

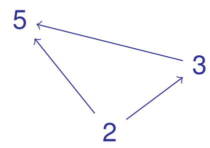
where 1 and 2 are two distinct objects different from x and y.

Relations and Orderings

Consider the familiar "strictly less than" relation $<$ on the set $\{2, 3, 5\}$. We have that

$$2 < 3, \quad 2 < 5, \quad 3 < 5,$$

which can be visualized as follows:



Each "arrow" in the picture can be represented by an ordered pair:

$$(2, 3), \quad (2, 5), \quad (3, 5).$$

The set of the above ordered pairs completely captures the information of the above "strictly less than" relation:

$$R = \{(2, 3), (2, 5), (3, 5)\}.$$

In general:

Definition 3.2

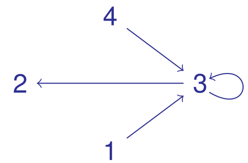
A relation R is a set of ordered pairs.

If R is a relation and $(x, y) \in R$, then we sometimes write xRy.

Example 3.1: Let $\omega = \{0, 1, 2, 3, \dots\}$ be the set of all natural numbers.

- The usual “strictly less than” relation $<$ on ω is defined formally to be the set:
$$<= \{(x, y) \in \omega \times \omega \mid x \text{ is strictly less than } y\}.$$
For instance, $(0, 1) \in <$ or $0 < 1$.
- The divisibility relation $|$ on ω is defined to be the set
$$| = \{(m, n) \in \omega \times \omega : \exists k \in \omega (m \cdot k = n)\}.$$
For instance, $3 \mid 9$, $5 \mid 10$, etc.
- The identity relation id on ω is defined to be the set
$$\mathrm{id} = \{(n, n) \mid n \in \omega\}$$

Example 3.2: The set $R = \{(1, 3), (4, 3), (3, 3), (3, 2)\}$ of ordered pairs is a relation.



Example 3.3: The membership relation $\in$ on the set $\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\}$ is the set
$$\in = \{(\emptyset, \{\emptyset\}), (\{\emptyset\}, \{\{\emptyset\}\})\}.$$

Example 3.4: Let X be a fixed nonempty set. The strict inclusion relation $\subset$ on subsets of X is the set
$$\subset = \{(A, B) \mid A \subseteq B \subseteq X,\ A \neq B\}.$$

Note: In particular, the empty set $\emptyset$ is a relation (called the *empty relation*).

Lemma 3D

If $(x, y) \in R$, then $x, y \in \bigcup\bigcup R$.

Proof. $(x, y) \in R \implies \{\{x\}, \{x, y\}\} \in R \implies \{\{x\}, \{x, y\}\} \subseteq \bigcup R$

(since $a \in A \implies a \subseteq \bigcup A$)

$\implies \{x, y\} \in \bigcup R \implies \{x, y\} \subseteq \bigcup\bigcup R \implies x, y \in \bigcup\bigcup R$ □

Definition 3.3

We define the domain of R ($\mathrm{dom}\, R$), the range of R ($\mathrm{ran}\, R$), and the field of R ($\mathrm{fld}\, R$) as
$$\mathrm{dom}\, R = \{x \in \bigcup\bigcup R \mid \exists y\, (x, y) \in R\},$$
$$\mathrm{ran}\, R = \{y \in \bigcup\bigcup R \mid \exists x\, (x, y) \in R\},$$
$$\mathrm{fld}\, R = \mathrm{dom}\, R \cup \mathrm{ran}\, R.$$
If $\mathrm{fld}\, R \subseteq A$, then we say that R is a relation on A.

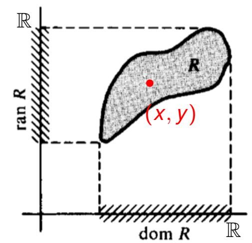
Example 3.5: For the relation $R = \{(1, 3), (4, 2), (3, 3), (3, 2)\}$, we have that $\mathrm{dom}\, R = \{1, 4, 3\}$, $\mathrm{ran}\, R = \{3, 2\}$, $\mathrm{fld}\, R = \{1, 2, 3, 4\}$.

Note: By Separation Axiom, $\mathrm{dom}\, R$, $\mathrm{ran}\, R$ and $\mathrm{fld}\, R$ are all sets.

Example 3.6: For the relation
$$< = \{(x, y) \in \omega \times \omega \mid x \text{ is strictly less than } y\},$$
$\mathrm{dom}\, < = \omega$, $\mathrm{ran}\, < = \omega \setminus \{0\}$, $\mathrm{fld}\, < = \omega$, and $<$ is a relation on ω.

Example 3.7: Let X be a fixed nonempty set. For the strict inclusion relation $\subset = \{(A, B) \mid A \subseteq B \subseteq X,\ A \neq B\}$, we have $\mathrm{fld} \subset = \wp(X)$, since $\emptyset \subset A$ for any nonempty $A \subseteq X$.

Example 3.8: A set $R \subseteq \mathbb{R} \times \mathbb{R}$ is a relation. R can be viewed as a subset of the coordinate plane. The projection of R onto the horizontal axis is dom R, and the projection onto the vertical axis is ran R.



We can define an ordered triple as

$$\begin{aligned}(x,y,z) &= ((x,y),z)\\ &= \{\{(x,y)\},\{(x,y),z\}\}\\ &= \{\{\{\{x\},\{x,y\}\}\},\{\{\{x\},\{x,y\}\},z\}\}\end{aligned}$$

Similarly, an ordered quadruple is defined as

$$\begin{aligned}(x_1,x_2,x_3,x_4) &= ((x_1,x_2,x_3),x_4)\\ &= (((x_1,x_2),x_3),x_4)\end{aligned}$$

Continue in this way, an ordered n-tuple is defined as

$$\begin{aligned}(x_1,x_2,\dots,x_n) &= ((x_1,x_2,\dots,x_{n-1}),x_n)\\ &= (((x_1,x_2,\dots,x_{n-2}),x_{n-1}),x_n)\\ &= \dots\\ &= ((\dots((x_1,x_2),x_3),\dots),x_n)\end{aligned}$$

In particular, we stipulate that a 1-tuple $(x) = x$.

Let $A_1, A_2, \dots, A_n$ be sets. Define

- $$\begin{aligned}A_1 \times A_2 \times A_3 &= (A_1 \times A_2) \times A_3\\ &= \{(x_1,x_2,x_3) : x_1 \in A_1,\ x_2 \in A_2 \text{ and } x_3 \in A_3\}\end{aligned}$$

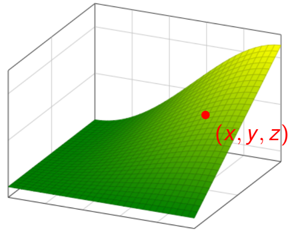
  $\vdots$

- $$\begin{aligned}A_1 \underbrace{\times \cdots \times}_{n} A_n &= (A_1 \underbrace{\times \cdots \times}_{n-1} A_{n-1}) \times A_n\\ &= \{(x_1,\dots,x_n) : x_1 \in A_1,\dots,x_n \in A_n\}\end{aligned}$$

Definition 3.4

An n-ary relation R on A is a set of ordered n-tuples with all components in A, that is, $R \subseteq A \underbrace{\times \cdots \times}_{n} A$.

Example 3.9: The following picture visualizes a ternary relation R on $\mathbb{R}$.



Definition 3.5

Let R be a (binary) relation on a set A.

- *R is said to be reflexive if xRx for every $x \in A$.*
- *R is said to be irreflexive if xRx for no $x \in A$.*
- *R is said to be symmetric if for all $x, y \in A$*
  $$xRy \Longrightarrow yRx.$$
- *R is said to be transitive if for all $x, y, z \in A$*
  $$xRy \text{ and } yRz \Longrightarrow xRz.$$

Example 3.10: The "strictly less than" relation $<$ on ω is

- irreflexive, since $n < n$ does not hold for any $n \in \omega$;
- not reflexive, since it is irreflexive;
- not symmetric, since, e.g. $0 < 1$ but $1 \not< 0$;
- transitive, since $[n < m \text{ and } m < k] \Longrightarrow n < k$.

The "less than or equal to" relation $\leq$ on ω is

- reflexive, since $n \leq n$ for any $n \in \omega$;
- not irreflexive, since it is reflexive.

Example 3.11: The relation $R = \{(1,2),(2,2)\}$ on $\{1,2\}$ is neither reflexive nor irreflexive.

Example 3.12: The membership relation

$$\in = \{(\emptyset,\{\emptyset\}),(\{\emptyset\},\{\{\emptyset\}\})\}$$

on the set $\{\emptyset,\{\emptyset\},\{\{\emptyset\}\}\}$ is

- irreflexive, since $\emptyset \notin \emptyset$, $\{\emptyset\} \notin \{\emptyset\}$, $\{\{\emptyset\}\} \notin \{\{\emptyset\}\}$;
- not symmetric, since $\emptyset \in \{\emptyset\}$ while $\{\emptyset\} \notin \emptyset$;
- not transitive, since $\emptyset \in \{\emptyset\}$ and $\{\emptyset\} \in \{\{\emptyset\}\}$ while $\emptyset \notin \{\{\emptyset\}\}$.

Example 3.13: Let be a fixed nonempty set. The strict inclusion relation

$$\subset = \{(A,B) \mid A \subseteq B \subseteq X,\ A \neq B\}.$$

on $\wp(X)$ is

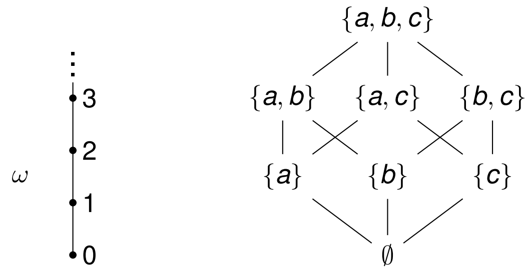
- irreflexive, since $A \not\subset A$ for any $A \subseteq X$;
- not symmetric, since $\emptyset \subset X$ but $X \not\subset \emptyset$;
- transitive, since $[A \subset B \subseteq X \text{ and } B \subset C \subseteq X] \Longrightarrow A \subset C \subseteq X$.

Definition 3.6

A relation R on a set A is called a partial ordering on A if R is transitive and irreflexive.

Example 3.14: The following are partial orderings:

- The “strictly less than” relation $<$ on ω.
- The strict inclusion relation $\subset$ on $\wp(\{a, b, c\})$.



Example 3.15: The membership relation $\in$ on $A = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\}$ is not a partial ordering, as it is not transitive. However, the membership relation $\in$ on $B = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$ is a partial ordering, since $\in$ is transitive on B.

We usually denote partial orderings by the symbol $<$, and define

$$x \leq y \text{ iff either } x < y \text{ or } y = x.$$

[Digression: In the study of partial orderings, there is always the question of whether to use strict orderings ($<$) or weak orderings ($\leq$) as the basic concept. “$<$” requires that a partial ordering be irreflexive, while “$\leq$” requires that a partial ordering on A be reflexive on A. Each alternative has its own minor advantages and disadvantages, see page 170 of the book for discussions.]

Theorem 7A

Assume that $<$ is a partial ordering on a set A. For arbitrary $x, y, z \in A$:

(a) *At most one of the following three alternatives holds:*

$$x < y, \quad x = y, \quad y < x.$$

(b) $x \leq y \leq x \Longrightarrow x = y$.

Proof. (a) If $x < y$ and $x = y$, then $x < x$, contradicting irreflexivity. If $x < y$ and $y < x$, then by transitivity, $x < x$, again contradicting irreflexivity.

(b) If $x \leq y \leq x$ and $x \neq y$, then $x < y < x$, which contradicts (a). □

Definition 3.7

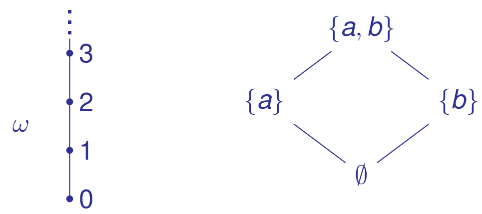
A relation R on a set A is called a linear ordering on A if R is transitive and satisfies trichotomy on A, i.e., for any $x, y \in A$, exactly one of the following three alternatives holds:

$$xRy, \quad x = y, \quad yRx.$$

For example:

- The “strictly less than” relation $<$ on ω is a linear ordering. (Trichotomy: for every $x, y \in \omega$, exactly one of the following three alternatives holds: $x < y$, $\quad x = y$, $\quad y < x$.)
- The strict inclusion relation $\subset$ on $\wp(\{a, b\})$ for $a \neq b$ is not a linear ordering, as it does not satisfy trichotomy: for $\{a\}, \{b\} \in \wp(\{a, b\})$,

$$\{a\} \not\subset \{b\}, \quad \{a\} \neq \{b\}, \quad \{b\} \not\subset \{a\}.$$



Theorem 3R

Let R be a linear ordering on a set A. Then
(i) R is connected, i.e., for distinct $x, y \in A$, either xRy or yRx.
Thereby, R is a linear ordering iff R is transitive and connected.
(ii) R is irreflexive, i.e., xRx for no $x \in A$. Thereby R is a partial ordering.

Proof. (i) Obvious by trichotomy, since $x \neq y$ for distinct $x, y \in A$.
(ii) For any $x \in A$, since $x = x$, it follows from trichotomy that xRx does not hold. □

It follows that linear orderings R can not have cycles such as

$$x_1 R x_2, \quad x_2 R x_3, \quad x_3 R x_4, \quad x_4 R x_1.$$

Because if the above cycle exists, then by transitivity $x_1 R x_1$, contradicting the irreflexivity.

Definition 3.8

Let $<$ be a partial ordering on a set A.

- *An element $a \in A$ is called a minimal element of A with respect to $<$ if for every $x \in A$,*
$$x \leq a \Longrightarrow x = a.$$
- *An element $a \in A$ is called a maximal element of A with respect to $<$ if for every $x \in A$,*
$$a \leq x \Longrightarrow x = a.$$

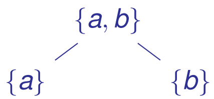
Definition 3.9

Let $<$ be a partial ordering on a set A.

- *An element $a \in A$ is called the least element or minimum of A with respect to $<$, denoted by $\min A$, if*
$$a \leq x \text{ for every } x \in A.$$
- *An element $a \in A$ is called the greatest element or maximum of A with respect to $<$, denoted by $\max A$, if*
$$x \leq a \text{ for every } x \in A.$$

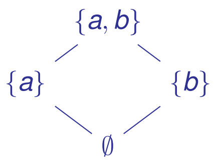
For example:

- Consider the strict inclusion relation $\subset$ on the set $A = \{\{a, b\}, \{a\}, \{b\}\}$.



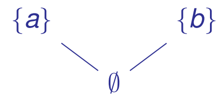
The minimal elements of A are $\{a\}$ and $\{b\}$, but $\min A$ does not exist. Both the maximal element and the greatest element of A are $\{a, b\}$.

- Consider the strict inclusion relation $\subset$ on the set $B = \{\{a, b\}, \{a\}, \{b\}, \emptyset\}$.



Both the minimal element and the least element of B are $\emptyset$. Both the maximal element and the greatest element of B are $\{a, b\}$.

- Consider the strict inclusion relation $\subset$ on the set $C = \{\{a\}, \{b\}, \emptyset\}$.



Both the minimal element and the least element of C are $\emptyset$. The maximal elements of C are $\{a\}$ and $\{b\}$, but $\max C$ does not exist.

The least (greatest) element of A (if exists) must be a minimal (maximal) element of A. But the converse is not true in general.

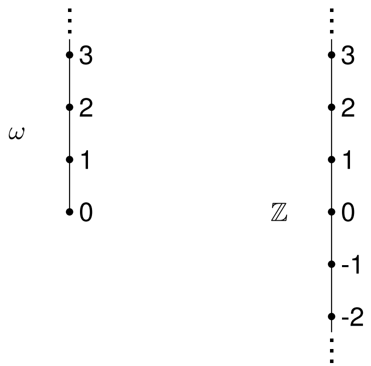
For linear orderings, the concept of least (greatest) element coincides with that of minimal (maximal) element.

A set A can have at most one least (greatest) element, since if both a and b are least (greatest) elements of A, then by definition, $a \leq b$ and $b \leq a$, thus $a = b$.

Definition 3.10

A linear ordering R on a set A is said to be a well ordering if every nonempty subset of A has a least element.

For example, The “strictly less than” relation $<$ on ω is a well ordering (a rigorous proof will be given in Chapter 4), but $<$ on $\mathbb{Z}$ is not a well ordering, as, e.g., $\mathbb{Z}$ does not have a least element.



Equivalence Relations

Definition 3.11

A relation R on a set A is called an equivalence relation if R is reflexive, symmetric and transitive.

Example 3.16: On $\mathbb{R}$:

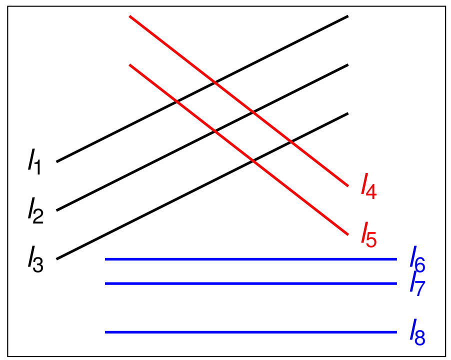
- The identity relation $=$ is an equivalence relation;
- The “strictly less than” relation $<$ is transitive, but it is not reflexive or symmetric, thus not an equivalence relation.
- The relation $\equiv$ defined by

  $$x \equiv y \iff |x| = |y|$$

  is an equivalence relation.
  (Transitivity: $x \equiv y$ and $y \equiv z \implies |x| = |y| = |z| \implies x \equiv z$)

Example 3.17: Let $S = \{l_1, l_2, l_3, l_4, l_5, l_6, l_7, l_8\}$ be the set of 8 lines in the following picture, and $\|$ the parallel relation on S, that is

$$\| = \{(l_i, l_j) \in S \times S \mid l_i \text{ is parallel to } l_j\}.$$



Then $\|$ is an equivalence relation on S, since
reflexivity: (we stipulate that) $l_i \| l_i$ for all $l_i \in S$;
symmetricity: $l_i \| l_j \implies l_j \| l_i$;
transitivity: $l_i \| l_j$ and $l_j \| l_k \implies l_i \| l_k$.

Example 3.18: Let $\equiv_3$ be a relation on $\mathbb{Z}$ defined by

$$x \equiv_3 y \iff 3 \mid x - y,$$

i.e., $x \equiv_3 y$ iff x has the same remainder as y when divided by 3, or x is congruent to y modulo 3. Show that $\equiv_3$ is an equivalence relation.

Proof. **Reflexivity:** For any $x \in \mathbb{Z}$, clearly, $3 \mid 0$, i.e., $3 \mid x - x$ or $x \equiv_3 x$.

Symmetricity: For any $x, y \in \mathbb{Z}$,

$$x \equiv_3 y \Longrightarrow 3 \mid x - y \Longrightarrow 3 \mid y - x \Longrightarrow y \equiv_3 x.$$

Transitivity: For any $x, y, z \in \mathbb{Z}$,

$$\begin{aligned} x \equiv_3 y \text{ and } y \equiv_3 z &\Longrightarrow 3 \mid x - y \text{ and } 3 \mid y - z \\ &\Longrightarrow 3 \mid (x - y) + (y - z) \\ &\Longrightarrow 3 \mid x - z \\ &\Longrightarrow x \equiv_3 z. \end{aligned}$$

□

Definition 3.12

Let $\equiv$ be an equivalence relation on a set A. Given $a \in A$. The set

$$[a]_\equiv = \{x \in A : x \equiv a\}$$

is called an equivalence class of a (modulo $\equiv$). If the relation $\equiv$ is clear from the context, we may only write $[a]$.

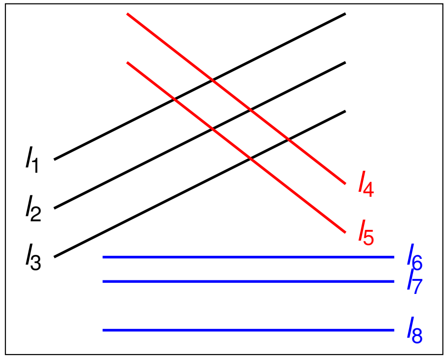
The element a is called a representative of the equivalence class $[a]$.

Example 3.19: Let $\equiv$ be the equivalence relation on $\mathbb{R}$ defined by $x \equiv y$ iff $|x| = |y|$. For any $r \in \mathbb{R}$,

$$[r] = \{x \in \mathbb{R} : x \equiv r\} = \{x \in \mathbb{R} : |x| = |r|\} = \{r, -r\}.$$

Example 3.20: Let $S = \{l_1, l_2, l_3, l_4, l_5, l_6, l_7, l_8\}$ be the set of 8 lines in the following picture, and $\parallel$ the parallel relation on S, that is

$$\parallel = \{(l_i, l_j) \in S \times S \mid l_i \text{ is parallel to } l_j\}.$$



Then

$$[l_1] = [l_2] = [l_3] = \{l_1, l_2, l_3\},$$
$$[l_4] = [l_5] = \{l_4, l_5\},$$
$$[l_6] = [l_7] = [l_8] = \{l_6, l_7, l_8\}.$$

Example 3.21: Let $\equiv_3$ be the relation "has the same remainder when divided by 3 as" on $\mathbb{Z}$. For any $k \in \mathbb{Z}$,

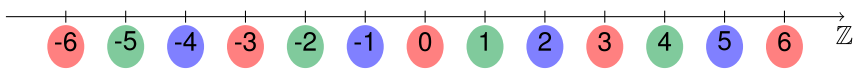
$$[k] = \{x \in \mathbb{Z} : 3 \mid k - x\}.$$

E.g.,
$$\begin{aligned} [0] &= \{0, 3, -3, 6, -6, \dots\} = \{3k : k \in \mathbb{Z}\} \\ [1] &= \{1, -2, 4, -5, \dots\} = \{3k + 1 : k \in \mathbb{Z}\} \\ [2] &= \{2, -1, 5, -4, \dots\} = \{3k + 2 : k \in \mathbb{Z}\} \end{aligned}$$

$$\begin{aligned} [0] &= [3] = [-3] = \dots \\ [1] &= [-2] = [4] = \dots \\ [2] &= [-1] = [5] = \dots \end{aligned}$$

It is easy to check that

$$[0] \cap [1] = \emptyset, \quad [1] \cap [2] = \emptyset, \quad [0] \cap [2] = \emptyset.$$



Proposition 3.13

Let $\equiv$ be an equivalence relation on a set A. For any $x, y \in A$,

(i) $x \in [x]$;

(ii) *If* $y \in [x]$, *then* $[y] = [x]$;

(iii) *If* $[x] \neq [y]$, *then* $[x] \cap [y] = \emptyset$.

Proof. (i) As $\equiv$ is reflexive, $x \equiv x$, which implies that

$$x \in \{z \in A : z \equiv x\} = [x].$$

(ii) Assume $y \in [x]$, i.e., $y \equiv x$. By transitivity and symmetricity of the equivalence relation $\equiv$, we have that, for any $a \in A$,

$$a \in [y] \iff a \equiv y \overset{y \equiv x}{\iff} a \equiv x \iff a \in [x],$$

and thus $[x] = [y]$.

(iii) If $z \in [x] \cap [y]$, then $[x] = [z] = [y]$ by (ii). □

Definition 3.14

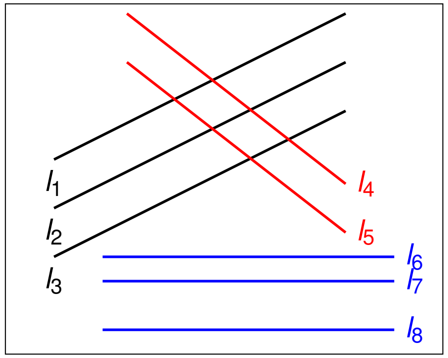
Let $\equiv$ be an equivalence relation on a set A. The set of all equivalence classes of $\equiv$ is called the quotient set of A by $\equiv$, denoted by $A/_{\equiv}$. That is

$$A/_{\equiv} = \{[x] : x \in A\}.$$

Example 3.22: Let $\equiv$ be an equivalence relation on $\mathbb{R}$ defined by $x \equiv y$ iff $|x| = |y|$. Then

$$\mathbb{R}/_{\equiv} = \{[r] : r \in \mathbb{R}\} = \{\{r, -r\} : r \in \mathbb{R}\}.$$

Example 3.23: Let $S = \{l_1, l_2, l_3, l_4, l_5, l_6, l_7, l_8\}$ be the set of 8 lines in the following picture, and $\|$ the parallel relation on S.



Then

$$S/_{\|} = \{[l_1], [l_4], [l_6]\} = \{\{l_1, l_2, l_3\}, \{l_4, l_5\}, \{l_6, l_7, l_8\}\}.$$

Example 3.24: Let $\equiv_3$ be the relation "has the same remainder when divided by 3 as" on $\mathbb{Z}$.

$$\mathbb{Z}/_{\equiv_3} = \{[k] : k \in \mathbb{Z}\} = \{\, [0],\ [1],\ [2] \,\}.$$

$$[0] = \{0, 3, -3, 6, -6, \dots\}$$

$$[1] = \{1, -2, 4, -5, \dots\}$$

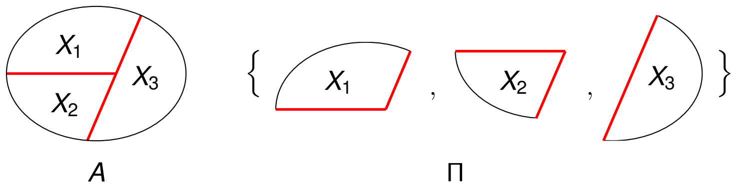
$$[2] = \{2, -1, 5, -4, \dots\}$$

Definition 3.15

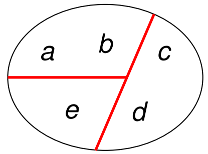
Let A be a non-empty set. A set Π of nonempty subsets of A is called a partition of A if

- *for any $X, Y \in \Pi$, if $X \neq Y$, then $X \cap Y = \emptyset$ (i.e., elements of Π are pairwise disjoint);*
- *$A = \bigcup_{X \in \Pi} X$ (i.e., Π is exhaustive);*

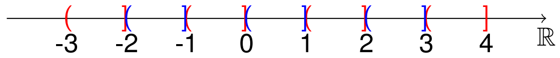
Intuitively, the above definition says that, if a set A is partitioned into some pairwise disjoint non-empty subsets, then we call the set Π consisting of all these subsets a partition of A.



Example 3.25: The set $\Pi = \{\{a,b\},\{c,d\},\{e\}\}$ is a partition of the set $\{a,b,c,d,e\}$.



Example 3.26: The set $\{(r-1, r] : r \in \mathbb{Z}\}$ is a partition of $\mathbb{R}$



Example 3.27: The quotient set

$$\mathbb{Z}/_{\equiv_3} = \{\,[0], [1], [2]\,\}$$

is a partition of $\mathbb{Z}$. This result is not incidental.