Chapter 1-2: Introduction, Axioms and Operations

January 14 &16, 2014

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Chapter 1: Introduction

A *set* *is a collection into a whole of definite, distinct objects of our intuition or our thought. The objects are called* *elements* *(or* *members**) of the set.*

—- *Georg Cantor, the founder of set theory*

For example, the following collections can be considered sets:

{  ,  ,  }

{  ,  ,  ,  }

$\mathbb{N} = \{0, 1, 2, 3, \ldots\}$, $A = \left\{\sqrt{2}, \frac{2}{7}, \sin 20^\circ\right\}$, $X = \{\{a\}, b\}$, etc.

Let A be a set, and x some object.

- If x is a element of A, then we write $x \in A$
- If x is not a element of A, then we write $\neg(x \in A)$ or $x \notin A$

For example:

- $x \in \{x, y, z\}$.
- $-5 \notin \{0, 1, 2, 3\}$.
-  $\in$ {  ,  ,  ,  }
-  $\in$ {  , b }
- $\{a\} \in \{ \{a\}, b \}$

Principle of Extensionality

If two sets A and B have exactly the same elements, then they are equal, denoted by $A = B$. Otherwise, we write $\neg(A = B)$ or $A \neq B$.

That is, if for every object x,

$$x \in A \iff x \in B,$$

then $A = B$.

From this principle, we know the following:

- If $A = \{\sqrt{2}, -\sqrt{2}\}$ and B is the set of all solutions to the equation
$$x^2 = 2,$$
then $A = B$.
- A set cannot have duplicate elements, that is, e.g.
$$\{a, a, b\} = \{a, b\}.$$
- Elements of a set do not have any order, that is, e.g.
$$\{a, b\} = \{b, a\}.$$

Usually, we represent a set in the following two ways:

1. List the elements of the set. For example,
   - $A = \{a, b, c\}$ is the set consisting of three elements a, b, c.
   - $K = \{0, 2, 4, 6, 8, \dots\}$ is the set of all even natural numbers.
2. Present the precise condition of an object being an element of the set. For example, the above set K can also be represented as
$$K = \{n \mid n \text{ is an even natural number}\}$$
or
$$K = \{n \in \mathbb{N} : n \text{ is even}\}.$$

- $X = \{x \in \mathbb{Z} \mid x + 1 = 0\} = \{-1\}$.

We call a set which has exactly one element a *singleton*.

For example, $\{a, b, c\}$ is a set having 3 elements; $\{\{a, b, c\}\}$ is a singleton.

- Let $Y = \{x \in \mathbb{N} \mid x + 1 = 0\}$. Clearly, Y has no elements at all.

A set which has no elements is called an *empty set*, denoted by $\emptyset$ or $\varnothing$.

By Principle of Extensionality, the empty set is unique.

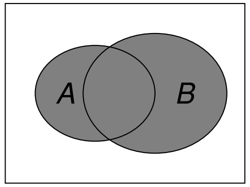
The empty set may seem to be useless, but in fact, it is very important. Using set-theoretic operations, many sets can be constructed from $\emptyset$.

- $\{\emptyset\}$ is a singleton consisting of the unique element $\emptyset$. Note that $\{\emptyset\} \neq \emptyset$, since $\emptyset \in \{\emptyset\}$ and $\emptyset \notin \emptyset$.
- $\{\{\emptyset\}\}$ is a singleton consisting of the unique element $\{\emptyset\}$, and $\{\{\emptyset\}\} \neq \{\emptyset\}$.
- The following are distinct singletons:
$$\{\{\{\emptyset\}\}\}, \ \{\{\{\{\emptyset\}\}\}\}, \ \{\{\{\{\{\emptyset\}\}\}\}\}, \ \dots$$
- The set $\{\emptyset, \{\emptyset\}\}$ has two elements, namely $\emptyset$ and $\{\emptyset\}$.

For any sets A and B, the set whose elements are those belonging either to A or to B (or both) is called the union of A and B, denoted by $A \cup B$. That is

$$A \cup B = \{x \mid x \in A \vee x \in B\}.$$

$A \cup B$



Venn Diagram

For example:

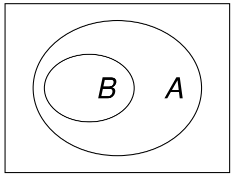
- If $A = \{a, b\}$ and $B = \{b, c, d\}$, then $A \cup B = \{a, b, c, d\}$.
- $\{\emptyset\} \cup \{\{\emptyset\}\} = \{\emptyset, \{\emptyset\}\}$.
- $\emptyset \cup A = A$, $A \cup A = A$, for any set A.

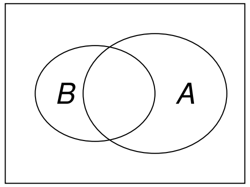
We say that a set B is a subset of a set A (or B is included in A), denoted by $B \subseteq A$, if every element of B is an element of A. That is

$$B \subseteq A \iff \forall x (x \in B \to x \in A).$$

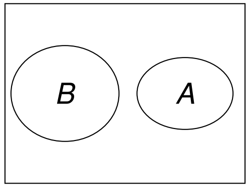
If B is not a subset of A, then we write $B \not\subseteq A$.

$B \subseteq A$





$B \not\subseteq A$



For example:

- Let $A = \{a, b, c, d\}$, $B = \{a, d\}$, $C = \{c, d\}$. Then $B \subseteq A$, $B \not\subseteq C$.
- $\{\emptyset\} \subseteq \{\emptyset, \{\emptyset\}\}$, $\quad \emptyset \subseteq \{\emptyset, \{\emptyset\}\}$.
- For any sets A and B,
$$A \subseteq A, \quad \emptyset \subseteq A, \quad A \subseteq A \cup B.$$
- If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$. (Transitivity)

By Principle of Extensionality,

$$A = B \quad \text{iff} \quad A \subseteq B \text{ and } B \subseteq A.$$

If $B \subseteq A$ and $B \neq A$, then we call B a *proper subset* of A and write $B \subset A$ or $B \subsetneq A$.

For example,

- if $A = \{\emptyset, \{\emptyset\}\}$, $B = \{\emptyset\}$, then $B \subset A$.
- $\emptyset \subset A$ for any $A \neq \emptyset$.

Given a set A. The set of all subsets of A is called the power set of A, denoted by $\wp(A)$ or $\wp A$. That is,

$$\wp(A) = \{X \mid X \subseteq A\}.$$

For example,

- If $A = \{a, b\}$, then
$$\wp(A) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}.$$
- $\wp(\emptyset) = \{X : X \subseteq \emptyset\} = \{\emptyset\}$.
- $\wp(\{\emptyset\}) = \{X : X \subseteq \{\emptyset\}\} = \{\emptyset, \{\emptyset\}\}$.
- $\wp(\{\emptyset, \{\emptyset\}\}) = \{X : X \subseteq \{\emptyset, \{\emptyset\}\}\} = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$.

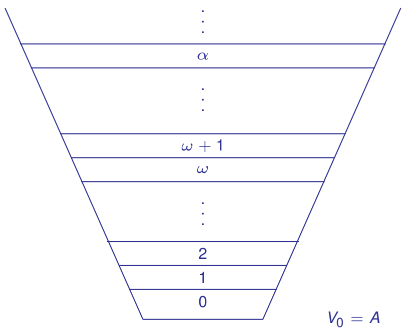
Constructing sets

- $V_0 = \{a, b, c, d, \ldots\}$, the set A of atoms
- $V_1 = V_0 \cup \wp(V_0) = A \cup \wp(A)$
- $V_2 = V_1 \cup \wp(V_1)$

  $\vdots$
- $V_{n+1} = V_n \cup \wp(V_n)$

  $\vdots$
- $V_\omega = V_1 \cup V_2 \cup \cdots$
- $V_{\omega+1} = V_\omega \cup \wp(V_\omega)$

  $\vdots$



$V_0 \subseteq V_1 \subseteq V_2 \subseteq \ldots$

$0, 1, 2, \ldots, \omega, \omega+1, \omega+2, \ldots, \alpha, \ldots$

ordinal numbers

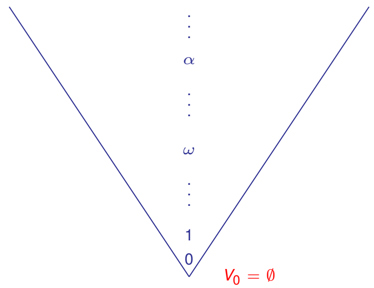
Constructing sets

- $V_0 = \emptyset$
- $V_1 = V_0 \cup \wp(V_0) = \wp(\emptyset) = \{\emptyset\}$
- $V_2 = V_1 \cup \wp(V_1) = \{\emptyset\} \cup \wp(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}$

  $\vdots$
- $V_{n+1} = V_n \cup \wp(V_n)$

  $\vdots$
- $V_\omega = V_1 \cup V_2 \cup \cdots$
- $V_{\omega+1} = V_\omega \cup \wp(V_\omega)$

  $\vdots$



Is there a "set of all sets"?

We will prove later that such a collection is *too large* to be called a set. Informally, any collection of sets is called a *class*, but a class A is called a *set* only if it is included in some level V_α of the hierarchy of sets; otherwise A is called a *proper class*. The "collection of all sets" is a proper class.

Consider the class $C = \{x \mid x \notin x\}$.

Question: Is $C \in C$?

- If YES, then $C \notin C$. Hence $C \in C \Longrightarrow C \notin C$.
- If NO, then $C \in C$. Hence $C \notin C \Longrightarrow C \in C$.

—- Russell's Paradox

To avoid "set of all sets" and paradoxes, we will re-examine the definition of a set.

Since early 1900's, mathematicians took the axiomatic approach to sets and developed the so-called "axiomatic set theory". One of the common axiom systems is the Zermelo-Fraenkel system with Axiom of Choice (ZFC).

Chapter 2: Axioms and Operations

An *axiom* or postulate is a premise so evident as to be accepted as true without controversy. Axioms serve as a starting point of reasoning. For example, Euclidean plane geometry has five evident axioms or postulates, all theorems and results of lines, triangles, circles, etc. can be derived from these five axioms.

In this course, we will discuss the axioms of ZFC system. From these axioms, the theorems of set theory will follow.

It is sometimes said that "mathematics can be embedded in set theory", meaning that mathematical objects (such as numbers, differentiable functions) can be defined to be certain sets, and the theorems of mathematics then can be viewed as statements about sets. Therefore, these theorems will be provable from the axioms of set theory.

In order to state the axioms and give proofs rigorously, we will use *formulas*.

In our context, the simplest formulas are expressions of the forms:

$$a \in A \quad \text{and} \quad a = b.$$

More complicated formulas can be built up from these using the expressions

$$\forall x,\ \exists x,\ \neg,\ \wedge,\ \vee,\ \rightarrow,\ \leftrightarrow$$

together with parentheses to avoid ambiguity. That is, from formulas ϕ and ψ, we can construct longer formulas

$$\forall x\phi,\ \exists x\phi,\ \neg\phi,\ \phi\wedge\psi,\ \phi\vee\psi,\ \phi\rightarrow\psi,\ \phi\leftrightarrow\psi, \text{ etc.}$$

For example, the following are formulas:

- $x \in A \vee x \in B$
- $\forall x(x \notin B)$
- $\forall x(x \in A \leftrightarrow x \in B)$
- $A \subseteq B$ is not a formula, but it can be viewed as a shorthand for $\forall x(x \in A \rightarrow x \in B)$.

Extensionality Axiom

For any two sets A, B, if they have exactly the same elements, then they are equal:

$$\forall A \forall B(\forall x(x \in A \leftrightarrow x \in B) \rightarrow A = B).$$

Since

$$B \subseteq A \quad \text{iff} \quad \forall x(x \in B \rightarrow x \in A),$$

the Extensionality Axiom can be rephrased as

$$A = B \quad \text{if} \quad A \subseteq B \text{ and } B \subseteq A.$$

Empty Set Axiom

There is a set having no elements:

$$\exists B \forall x(x \notin B).$$

Definition 2.1

A set which has no elements is called an Empty Set, denoted by $\emptyset$ or $\varnothing$.

- By Empty Set Axiom, the empty set exists.
- By Extensionality Axiom, the empty set is unique.
- $\emptyset = \{x \mid x \neq x\}$

Pairing Axiom

For any sets u and v, there is a set having just u and v as elements:

$$\forall u \forall v \exists B \forall x \big(x \in B \leftrightarrow (x = u \lor x = v)\big).$$

Definition 2.2

For any sets u and v, the set whose elements are u and v is called a pair set, denoted by $\{u, v\}$. That is,

$$\{u, v\} = \{x \mid x = u \lor x = v\}.$$

- Given any set x, by Pairing Axiom the set $\{x, x\}$ exists. We write $\{x\}$ for $\{x, x\}$, and call it a *singleton*.
- Since $\emptyset$ is a set, by Pairing Axiom, $\{\emptyset, \emptyset\} = \{\emptyset\}$ is a set.
- Since $\emptyset$ and $\{\emptyset\}$ are sets, $\{\emptyset, \{\emptyset\}\}$ is a set.

Union Axiom (Preliminary Form)

For any sets A and B, there is a set whose elements are those sets belonging either to A or to B (or both):

$$\forall A \forall B \exists C \forall x \big(x \in C \leftrightarrow (x \in A \lor x \in B)\big).$$

Definition 2.3

For any sets A and B, the set whose elements are those sets belonging either to A or to B (or both) is called the union of A and B, denoted by $A \cup B$. That is,

$$A \cup B = \{x \mid x \in A \lor x \in B\}$$

For example, $\{\emptyset\} \cup \{\{\emptyset\}\} = \{\emptyset, \{\emptyset\}\}$ is a set.

Given sets x_1, x_2, x_3, by Pairing Axiom, $\{x_1, x_2\}$ and $\{x_3\}$ are sets. By Union Axiom, $\{x_1, x_2\} \cup \{x_3\}$ is a set, and we define

$$\{x_1, x_2, x_3\} = \{x_1, x_2\} \cup \{x_3\}.$$

Similarly, we can define $\{x_1, x_2, x_3, x_4\}$, and so forth.

Power Set Axiom

For any set A, there is a set whose elements are exactly all of the subsets of A:

$$\forall A \exists \mathscr{B} \forall x (x \in \mathscr{B} \leftrightarrow x \subseteq A).$$

Definition 2.4

For any set A, the set of all subsets of A is called the power set of A, denoted by $\wp(A)$ or $\wp A$. That is,

$$\wp(A) = \{x \mid x \subseteq A\}.$$

Note: $x \in \wp(A) \iff x \subseteq A$.

Example 2.1: For any sets A and B, if $\wp(A) = \wp(B)$, then $A = B$.

Proof. If $\wp(A) = \wp(B)$, then

$$A \subseteq A \Longrightarrow A \in \wp(A) = \wp(B) \Longrightarrow A \in \wp(B) \Longrightarrow A \subseteq B.$$

Similarly, we get $B \subseteq A$. Hence $A = B$. □

- $\emptyset = \{x \mid x \neq x\}$
- $\{u, v\} = \{x \mid x = u \lor x = v\}$
- $A \cup B = \{x \mid x \in A \lor x \in B\}$
- $\wp(A) = \{x \mid x \subseteq A\}$

The above sets are all of the form $\{x \mid \phi(x)\}$.

For any formula $\phi(x)$, is

$$\mathrm{A} = \{x \mid \phi(x)\}$$

always a set?

No. For example, $\mathrm{V} = \{x \mid x = x\}$ is not a set, that is, the set of all sets does not exist.

In order for the class A to be called a set, it must be included in some V_α of the hierarchy. In fact, it is enough for A to be included in any set c, for then

$$\mathrm{A} \subseteq c \subseteq V_\alpha$$

for some α.

Separation Axioms (or Subset Axioms)

For any formula $\phi(x, y_1, \ldots, y_n)$ not containing B, any sets $c, a_1, \ldots, a_n$,

$$B = \{x \in c \mid \phi(x, a_1, \ldots, a_n)\}$$

is a set.

Clearly, $B \subseteq c$.

For example:

- Let $c = \{\{a\}, \{a, b\}, \{b, d, e\}\}$ be a set. Then the following are sets:

$$B_0 = \{x \in c \mid a \in x\} = \{\{a\}, \{a, b\}\},$$
$$B_1 = \{x \in c \mid a \notin x\} = \{\{b, d, e\}\}.$$

- Let $c = \{a, b\}$ be a set. Then

$$B = \{x \in \wp(c) \mid x \text{ is a singleton}\} = \{\,\{a\}, \{b\}\,\}$$

is a set, where "x is a singleton" is a shorthand for the formula

$$\forall y(y \subseteq x \to (y = x \lor y = \emptyset)),$$

or

$$\forall y\big(\forall z(z \in y \to z \in x) \to (y = x \lor y = \emptyset)\big).$$

Theorem 2A

There is no set to which every set belongs.

Proof. Let A be an arbitrary set. We construct a set B such that $B \notin A$.

By Separation Axiom, B defined as follows is a set:

$$B = \{x \in A \mid x \notin x\}.$$

If $B \in A$, then by the definition of B,

$$B \notin B \iff B \in B,$$

which is impossible. Hence $B \notin A$, as desired. □

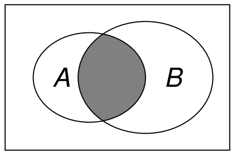
No set is an element of itself: $\forall x(x \notin x)$. [Follows from Regularity Axiom (Chapter 7)]

Definition 2.5

For any sets A and B, the intersection of A and B, denoted by $A \cap B$, is defined as

$$A \cap B = \{x \mid x \in A \land x \in B\}.$$

$A \cap B$



By Separation Axiom, we know that the intersection

$$A \cap B = \{x \mid x \in A \land x \in B\} = \{x \in A \mid x \in B\}$$

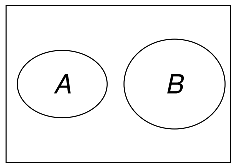
is a set.

For example:

- If $A = \{a, b, c\}$ and $B = \{b, c, d\}$, then $A \cap B = \{b, c\}$.
- $\{\emptyset\} \cap \{\emptyset, \{\emptyset\}\} = \{\emptyset\}$
- $\emptyset \cap A = \emptyset$, $A \cap A = A$ for any set A.
- $A \cap B \subseteq A$ for any sets A and B

Definition 2.6

If $A \cap B = \emptyset$, then we say that A and B are disjoint, or A does not intersect B.

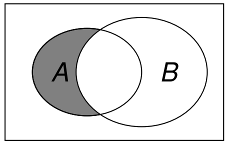


Definition 2.7

For any sets A and B, the relative complement of B in A or the difference of A and B, denoted by $A - B$ or $A \setminus B$, is defined as

$$A - B = \{x \mid x \in A \wedge x \notin B\}.$$

$A - B$



By Separation Axiom, we know that the difference

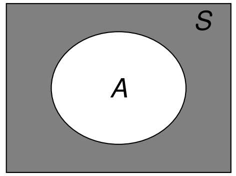
$$A - B = \{x \mid x \in A \wedge x \notin B\} = \{x \in A \mid x \notin B\}.$$

is a set.

Definition 2.8

Let A be a subset of S. The relative complement of S and A is called the complement of A in S, denoted by $\overline{A}$ or A^c or $-A$.

$\overline{A}$



For example, if $S = \{a, b, c, d\}$, $A = \{a, b, c\}$ and $B = \{b, c\}$, then

$$A - B = \{a\}, \quad B - A = \emptyset, \quad \overline{A} = \{d\}, \quad \overline{B} = \{a, d\}.$$

Some properties of difference and complement:

- $\overline{\overline{A}} = A$
- $\overline{\emptyset} = S, \quad \overline{S} = \emptyset$
- $A \cap \overline{A} = \emptyset, \quad A \cup \overline{A} = S$
- $A - B = A \cap \overline{B}$

Properties of operations on sets

Let A, B, C be sets.

Commutative laws $\quad A \cup B = B \cup A$ and $A \cap B = B \cap A$

Associative laws

$$A \cup (B \cup C) = (A \cup B) \cup C$$
$$A \cap (B \cap C) = (A \cap B) \cap C$$

Distributive laws

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

If $A \subseteq B$, then

$$A \cup C \subseteq B \cup C, \quad A \cap C \subseteq B \cap C.$$

Example 2.2: Show that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Proof. For any x, $\quad x \in A \cup (B \cap C) \Longrightarrow$

$$\begin{cases} x \in A \Longrightarrow & x \in A \cup B \text{ and } x \in A \cup C \Longrightarrow x \in (A \cup B) \cap (A \cup C) \\ \text{or} & \Uparrow \\ x \in B \cap C & \Longrightarrow x \in B \text{ and } x \in C \end{cases}$$

Hence, we conclude that $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$.

Conversely, for any x, if $x \in A$, then $x \in A \cup (B \cap C)$; if $x \notin A$, then

$$x \in (A \cup B) \cap (A \cup C) \Longrightarrow x \in A \cup B \text{ and } x \in A \cup C$$

$$\overset{x \notin A}{\Longrightarrow} x \in B \text{ and } x \in C \Longrightarrow x \in B \cap C \Longrightarrow x \in A \cup (B \cap C).$$

Hence $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$. □

Let A, B, C be subsets of a set S.

De Morgan's laws
$$C - (A \cup B) = (C - A) \cap (C - B)$$
$$C - (A \cap B) = (C - A) \cup (C - B)$$

Substituting S for C yields
$$\overline{(A \cup B)} = \overline{A} \cap \overline{B}$$
$$\overline{(A \cap B)} = \overline{A} \cup \overline{B}$$

Example 2.3: Show that $C - (A \cup B) = (C - A) \cap (C - B)$.

Proof. For any element x, we have that

$$\begin{aligned} x \in C - (A \cup B) &\iff x \in C \text{ and } x \notin A \cup B \\ &\iff x \in C \text{ and } (x \notin A \text{ and } x \notin B) \\ &\iff (x \in C \text{ and } x \notin A) \text{ and } (x \in C \text{ and } x \notin B) \\ &\iff x \in C - A \text{ and } x \in C - B \\ &\iff x \in (C - A) \cap (C - B). \end{aligned}$$

□

Example 2.4: Let A, B, C be subsets of a set S. Show that
$$A - (B - C) = (A - B) \cup (A \cap C).$$

Proof. We have that

$$\begin{aligned} A - (B - C) &= A \cap \overline{(B - C)} \quad (\text{since } X - Y = X \cap \overline{Y}) \\ &= A \cap \overline{(B \cap \overline{C})} \\ &= A \cap (\overline{B} \cup \overline{\overline{C}}) \quad (\text{by De Morgan's law}) \\ &= A \cap (\overline{B} \cup C) \\ &= (A \cap \overline{B}) \cup (A \cap C) \quad (\text{by distributive law}) \\ &= (A - B) \cup (A \cap C). \end{aligned}$$

□

Suppose we have an infinite collection of sets
$$\mathscr{A} = \{A_1, A_2, A_3, \ldots, A_n, \ldots\}.$$

By the preliminary form of Union Axiom, we know that the following finite unions are sets:
$$A_1 \cup A_2, \quad (A_1 \cup A_2) \cup A_3, \quad \ldots \quad A_1 \cup A_2 \cup \cdots \cup A_n, \quad \ldots$$

Definition 2.9

For any set $\mathscr{A}$, the union of $\mathscr{A}$, denoted by $\bigcup \mathscr{A}$, is defined as
$$\bigcup \mathscr{A} = \{x \mid x \text{ belongs to some element of } \mathscr{A}\} = \{x \mid \exists A \in \mathscr{A} (x \in A)\}.$$

To guarantee that $\bigcup \mathscr{A}$, particularly the infinite union $A_1 \cup A_2 \cup \cdots$, is a set, we need the general form of Union Axiom.

Union Axiom

For any set $\mathscr{A}$, there exists a set B whose elements are exactly the elements of the elements of $\mathscr{A}$:
$$\forall \mathscr{A} \exists B \forall x [x \in B \leftrightarrow \exists A \in \mathscr{A} (x \in A)].$$

For example:

- If $\mathscr{A} = \{A_1, A_2, \ldots, A_n, \ldots\}$, then $\bigcup \mathscr{A} = A_1 \cup A_2 \cup \cdots \cup A_n \cup \cdots$.
- $\bigcup \{\{a, b, c\}, \{c, d\}, \{c, e\}\} = \{a, b, c, d, e\}$
- $\bigcup \{a, b\} = a \cup b, \quad \bigcup \{a, b, c\} = a \cup b \cup c, \quad \ldots$
- $\bigcup \{a\} = a, \quad \bigcup \emptyset = \emptyset$
- If $\mathscr{A} = \{A_i \mid i \in I\}$, where I is referred to as an *index set*, we also write $\bigcup \mathscr{A} = \bigcup_{i \in I} A_i$.
- If $A \in \mathscr{A}$, then $x \in A \Longrightarrow x \in \bigcup \mathscr{A}$, thus $A \subseteq \bigcup \mathscr{A}$.

$\bigcup \wp(\{a,b\}) = \bigcup(\{\emptyset, \{a\}, \{b\}, \{a,b\}\}) = \emptyset \cup \{a\} \cup \{b\} \cup \{a,b\} = \{a,b\}$

Example 2.5: Show that for any set A,

$$\bigcup \wp(A) = A.$$

Proof. Since $A \in \wp(A)$, we have that $A \subseteq \bigcup \wp(A)$. Conversely, for any $x \in \bigcup \wp(A)$, by definition, there exists $X \in \wp(A)$ such that $x \in X$. Since $x \in X \subseteq A$, we obtain $x \in A$. □

Definition 2.10

For any nonempty set $\mathscr{A}$, the intersection of $\mathscr{A}$, denoted by $\bigcap \mathscr{A}$, is defined as

$$\bigcap \mathscr{A} = \{x \mid x \text{ belongs to every element of } \mathscr{A}\} = \{x \mid \forall A \in \mathscr{A}(x \in A)\}.$$

If $\mathscr{A} = \{A_1, A_2, \ldots, A_n, \ldots\}$, then $\bigcap \mathscr{A} = A_1 \cap A_2 \cap A_3 \cap \cdots$.

Since $\mathscr{A} \neq \emptyset$, there exists $C \in \mathscr{A}$, thus by Separation Axiom,

$$\bigcap \mathscr{A} = \{x \mid \forall A \in \mathscr{A}(x \in A)\} = \{x \in C \mid \forall A \in \mathscr{A}(x \in A)\}$$

is a set.

For example

- $\bigcap\{\{a,b,c\},\{c,d\},\{c,e\}\} = \{a,b,c\} \cap \{c,d\} \cap \{c,e\} = \{c\}$
- $\bigcap\{a,b\} = a \cap b, \quad \bigcap\{a,b,c\} = a \cap b \cap c, \quad \ldots$
- $\bigcap\{a\} = a. \quad \bigcap \emptyset$ is undefined or equal to some fixed set.
- If $\mathscr{A} = \{A_i \mid i \in I\}$, we also write $\bigcap \mathscr{A} = \bigcap_{i \in I} A_i$.

Example 2.6:

$$\bigcup \bigcap \{\{a\},\{a,b\}\} = \bigcup(\{a\} \cap \{a,b\}) = \bigcup\{a\} = a.$$

$$\bigcap \bigcup \{\{a\},\{a,b\}\} = \bigcap(\{a\} \cup \{a,b\}) = \bigcap\{a,b\} = a \cap b.$$

Example 2.7: If $\emptyset \neq \mathscr{A} \subseteq \mathscr{B}$, then $\bigcap \mathscr{B} \subseteq \bigcap \mathscr{A}$

Proof. For any $x \in \bigcap \mathscr{B}$, we show that $x \in \bigcap \mathscr{A}$, which is equivalent to showing that $x \in X$ for all $X \in \mathscr{A}$.

From $X \in \mathscr{A} \subseteq \mathscr{B}$, it follows that $X \in \mathscr{B}$, thereby $\bigcap \mathscr{B} \subseteq X$. Hence $x \in \bigcap \mathscr{B}$ implies that $x \in X$, as required. □

Distributive laws

$$A \cap \bigcup \mathscr{B} = \bigcup\{A \cap X \mid X \in \mathscr{B}\} = \bigcup_{X \in \mathscr{B}} (A \cap X)$$

$$A \cup \bigcap \mathscr{B} = \bigcap\{A \cup X \mid X \in \mathscr{B}\} = \bigcap_{X \in \mathscr{B}} (A \cup X)$$

De Morgan's laws (for $\mathscr{A} \neq \emptyset$)

$$\overline{\bigcup \mathscr{A}} = \bigcap\{\overline{X} \mid X \in \mathscr{A}\} = \bigcap_{X \in \mathscr{A}} \overline{X}$$

$$\overline{\bigcap \mathscr{A}} = \bigcup\{\overline{X} \mid X \in \mathscr{A}\} = \bigcup_{X \in \mathscr{A}} \overline{X}$$