Elements of Set Theory

Chapter 7: Orderings and Ordinals

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Definition 7.25

For any set A, define the cardinal number of A to be the least ordinal equinumerous to A, i.e., $|A| = \min\{\alpha \in \mathrm{Ord} \mid \alpha \approx A\}$.

For example, for the set ω, we have that

$$\omega \approx \omega + 1 \approx \omega + 2 \approx \cdots \approx \omega + \omega \approx \omega \cdot \omega \approx \dots$$

So, $\aleph_0 = |\omega| = \min\{\omega, \omega + 1, \omega + 2, \dots\} = \omega$.

Definition 7.26

An ordinal number α is called an initial ordinal if it is not equinumerous to any $\beta < \alpha$.

Clearly, an ordinal α is a cardinal iff α is an initial ordinal.

Example 7.21: ω is an initial ordinal, or a cardinal, but none of $\omega + 1, \omega + 2, \cdots, \omega + \omega, \omega \cdot \omega, \dots$ is an initial ordinal or a cardinal.

Recall: Let κ and λ be cardinals.

- Viewed as cardinals, we define $\kappa < \lambda$ iff $\kappa \preccurlyeq \lambda \wedge \kappa \neq \lambda$.
- Viewed as ordinals, we define $\kappa < \lambda$ iff $\kappa \in \lambda$

Fact 7.27

The ordering on cardinal numbers coincides with the ordering on ordinal numbers.

Proof. For any cardinals κ and λ, we show that

$$\kappa \in \lambda \iff \kappa \preccurlyeq \lambda \wedge \kappa \neq \lambda.$$

For "$\Longrightarrow$", we have that

$$\kappa \in \lambda \Longrightarrow \kappa \subset \lambda \Longrightarrow \kappa \preccurlyeq \lambda \wedge \kappa \neq \lambda.$$

Conversely,

$$\kappa \notin \lambda \Longrightarrow \lambda \underline{\in} \kappa \Longrightarrow \lambda \subseteq \kappa \Longrightarrow \lambda \preccurlyeq \kappa$$
$$\Longrightarrow \neg(\kappa \preccurlyeq \lambda \wedge \kappa \neq \lambda) \quad \text{(by C-S-B Theorem).}$$

□

Hartogs' Theorem

For any set A, there exists an ordinal α such that $\alpha \not\preccurlyeq A$. The least such ordinal, denoted by $h(A)$, is called the Hartogs' number of A.

Note: By the theorem and Axiom of Choice, $|A| < |h(A)|$.

Proof. Consider the set

$$W = \{(B, <) \mid B \subseteq A \wedge < \text{ is a well ordering on } B\}.$$

By Theorem 7.24, for each $(B, <) \in W$, there exists a unique ordinal β_B such that $(\beta_B, \in) \cong (B, <)$. By Replacement Axiom, H defined as follows is a set:

$$H = \{\beta_B \in \mathrm{Ord} \mid (\beta_B, \in) \cong (B, <) \text{ for some } (B, <) \in W\} \subsetneq \mathrm{Ord}.$$

We claim $\beta \in H$ for all $\beta \preccurlyeq A$, which implies that any ordinal $\alpha \notin H$ satisfies $\alpha \not\preccurlyeq A$.

Let $f : \beta \to A$ be a one-to-one function, and $B = \mathrm{ran}\, f \subseteq A$. Define an ordering $<$ on B as

$$<= \{(f(\gamma), f(\delta)) \mid \gamma \in \delta \in \beta\}.$$

Clearly, $(\beta, \in) \cong (B, <)$, thus $\beta \in H$.

We turn to showing that $h(A) = H$.

First, we show that H is an ordinal, which amounts to showing that H is transitive since H is a set of ordinals. Indeed,

$$\gamma \in \beta \in H \Longrightarrow \gamma \subseteq \beta \preccurlyeq A \Longrightarrow \gamma \preccurlyeq A \Longrightarrow \gamma \in H.$$

Second, we have that $H \not\preccurlyeq A$, as otherwise $H \in H$, which is impossible.

Lastly, H is the least such ordinal, since $\alpha \in H \Longrightarrow \alpha \preccurlyeq A$ by definition of H. □

Fact: $h(A)$ is an initial ordinal number, thus a cardinal number.

Proof. If $\beta \approx h(A)$ for some $\beta < h(A)$, then $\beta \preccurlyeq A$. It follows that $h(A) \approx \beta \preccurlyeq A$, contradicting $h(A) \not\preccurlyeq A$. □

- For the finite set $A = \{a_0, \ldots, a_{n-1}\}$, we have that
$$h(A) = |A| + 1 = n + 1.$$
In particular, $h(n) = n + 1$ for all $n \in \omega$.
- $\aleph_0 = \omega$ is the first infinite cardinal.
- By Hartogs' Theorem, $h(\omega)$ exists. Denote $\omega_1 = h(\omega) = \aleph_1$.
- Similarly, $\omega_2 = h(\omega_1) = \aleph_2$.

$\vdots$

- $\omega_\omega = \sup\{\omega_n \mid n \in \omega\} = \aleph_\omega$.
- $\omega_\omega + 1 = {\omega_\omega}^+ = \omega_\omega \cup \{\omega_\omega\}$
- $\omega_{\omega+1} = h(\omega_\omega) = \aleph_{\omega+1}$.

Rank

Axiom of Regularity (also known as Axiom of Foundation)

Every nonempty set A has an $\in$-minimal element, that is:
$$\forall A(A \neq \emptyset \rightarrow \exists m \in A(m \cap A = \emptyset)).$$

Note that the formula $m \cap A = \emptyset$ is equivalent to the formula
$$\forall a(a \in m \rightarrow a \notin A)$$
both of which characterize m being an $\in$-minimal element of A.

Theorem 7.28

There is no function f with domain ω such that
$$\cdots \in f(2) \in f(1) \in f(0).$$
In particular,

(a) *no set is a member of itself, i.e., $\forall x(x \notin x)$;*

(b) *there are no "cycles":*
$$x_0 \in x_1 \in \cdots \in x_n \in x_0.$$

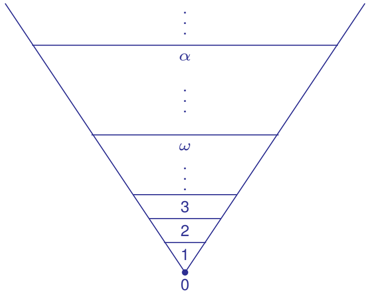
Proof. If there exists a function f such that $f(n^+) \in f(n)$ for each $n \in \omega$, then the set $\operatorname{ran} f = \{f(n) | n \in \omega\}$ has no $\in$-minimal element. This contradicts regularity axiom. □

- $V_0 = \emptyset$
- $V_1 = \wp V_0 = \wp\emptyset = \{\emptyset\}$
- $V_2 = \wp V_1 = \wp\{\emptyset\} = \{\emptyset, \{\emptyset\}\}$

  $\vdots$
- $V_{n+1} = \wp V_n$

  $\vdots$
- $V_\omega = \bigcup_{n\in\omega} V_n$
- $V_{\omega+1} = \wp V_\omega$

  $\vdots$



Formally, by transfinite recursion, we make the following definition:

Definition 7.29 (The Cumulative Hierarchy of Sets)

- $V_0 = \emptyset$
- $V_{\alpha+1} = \wp V_\alpha$
- $V_\lambda = \bigcup_{\alpha<\lambda} V_\alpha$ *if λ is a limit ordinal.*

Note: If $x \subseteq V_\alpha$, then $x \in V_{\alpha+1} = \wp V_\alpha$.

Lemma 7.30

1. *If $\beta \leq \alpha$, then $V_\beta \subseteq V_\alpha$,*
2. *V_α is transitive for each ordinal α.*

Proof. We prove by transfinite induction that the class

$$T = \{\alpha \in \mathrm{Ord} \mid \forall\beta < \alpha(V_\beta \subseteq V_\alpha) \wedge V_\alpha \text{ is transitive}\}$$

is equal to Ord. Hypothesize $\alpha \subseteq T$. We show that $\alpha \in T$.

- If $\alpha = 0$, then $\alpha \in T$, since $V_0 = \emptyset$ is transitive.
- If $\alpha = \gamma + 1$, then by induction hypothesis, $\forall\beta < \gamma(V_\beta \subseteq V_\gamma)$ and V_γ is transitive. Thus $\wp V_\gamma = V_{\gamma+1}$ is transitive, thereby $V_\gamma \subseteq V_{\gamma+1}$. For any $\beta < \gamma + 1 = \gamma \cup \{\gamma\}$, we have $\beta \underline{\in} \gamma$, thus $V_\beta \subseteq V_\gamma \subseteq V_{\gamma+1}$.
- If α is a limit ordinal and $\beta < \alpha$, then $V_\beta \subseteq \bigcup_{\gamma<\alpha} V_\gamma = V_\alpha$. To see that V_α is transitive, observe that by induction hypothesis, V_α is the union of set $\{V_\gamma \mid \gamma < \alpha\}$ of transitive sets, thus transitive. □

Definition 7.31

We say that a set x is grounded iff $x \subseteq V_\alpha$ for some ordinal α.

Note: Clearly, $x \subseteq V_\alpha$ iff $x \in V_{\alpha+1} = \wp V_\alpha$.

Call the class

$$V = \{x \mid x = x\}$$

of all sets the universal class or the universe.

Theorem 7.32

Every set is grounded, i.e., $V = \bigcup_{\alpha\in\mathrm{Ord}} V_\alpha$.

Proof. Assume towards a contradiction that some set c is not grounded. Then there exists a transitive set T that contains a ungrounded element (e.g. we can take T to be the transitive closure of the set $\{c\}$).

Consider the set

$$A = \{x \in T \mid x \text{ is not grounded}\}.$$

Since $A \neq \emptyset$, by Regularity Axiom, there exists some $m \in A$ such that $m \cap A = \emptyset$. We proceed to show that m is grounded, which will give the desired contradiction.

For any $x \in m$, we have $x \in T$ since T is transitive. But $x \notin A$, as $m \cap A = \emptyset$. This shows that x is grounded, i.e., $x \in V_{\alpha_x}$ for some ordinal α_x.

Now, by replacement axiom, $\{\alpha_x \mid x \in m\}$ is a set. Put

$$\gamma = \sup\{\alpha_x \mid x \in m\}.$$

Since $x \in V_{\alpha_x} \subseteq V_\gamma$ for all $x \in m$, it follows that

$$m \subseteq \bigcup_{x\in m} V_{\alpha_x} \subseteq V_\gamma,$$

namely, m is grounded. □

Definition 7.33 (rank)

The rank of a set x, denoted by $\operatorname{rank} x$, is defined to be the least ordinal α such that $x \subseteq V_\alpha$.

Equivalently,

$$\operatorname{rank} x = \alpha \text{ iff } \alpha \text{ is the least ordinal such that } x \in V_{\alpha+1}$$

Example 7.22: Recall:

$$V_0 = \emptyset, \quad V_1 = \wp V_0 = \{\emptyset\}$$

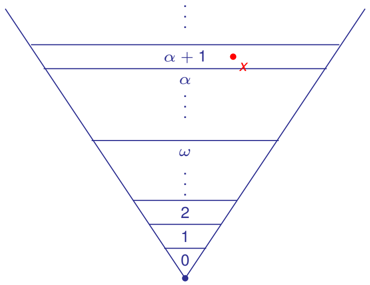
$$V_2 = \{\emptyset, \{\emptyset\}\}$$

$$V_3 = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$$

Thus

$$\operatorname{rank} \emptyset = 0, \quad \operatorname{rank} \{\emptyset\} = 1,$$

$$\operatorname{rank} \{\{\emptyset\}\} = 2, \quad \operatorname{rank} \{\emptyset, \{\emptyset\}\} = 2$$



Lemma 7.34

$V_\alpha = \{x \mid \operatorname{rank} x < \alpha\}$ for every ordinal α.

Proof. "$\supseteq$": If $\operatorname{rank} x = \beta < \alpha$, then $x \in V_{\beta+1} \subseteq V_\alpha$ (since $\beta + 1 \leq \alpha$).

"$\subseteq$": Suppose $x \in V_\alpha$. If $\alpha = \beta + 1$ for some β, then $x \in V_{\beta+1}$. Hence $\operatorname{rank} x \leq \beta < \alpha$ by the definition of rank.

If α is a limit ordinal, then

$$x \in V_\alpha = \bigcup_{\beta<\alpha} V_\beta \implies x \in V_\beta \text{ for some } \beta < \alpha$$

$$\implies x \subseteq V_\beta \text{ for some } \beta < \alpha.$$

Hence $\operatorname{rank} x \leq \beta < \alpha$. □

Theorem 7.35

(a) *If $a \in A$, then $\operatorname{rank} a < \operatorname{rank} A$.*

(b) *For any set A,*

$$\operatorname{rank} A = \sup\{\operatorname{rank} a + 1 \mid a \in A\}.$$

(c) *$\operatorname{rank} \alpha = \alpha$ for any ordinal α.*

Proof. (c) is left as an exercise. (a) Assume $a \in A$. By the definition of rank, we have that

$$A \subseteq V_{\operatorname{rank} A} \implies a \in V_{\operatorname{rank} A}$$

$$\implies \operatorname{rank} a < \operatorname{rank} A \quad \text{(by Lemma 7.34).}$$

(b) Put $B = \{\operatorname{rank} a + 1 \mid a \in A\}$. We show that $\operatorname{rank} A$ is the least upper bound of B.

First, we show that $\operatorname{rank} A$ is an upper bound of B. Indeed, for any a,

$$a \in A \implies \operatorname{rank} a < \operatorname{rank} A \quad \text{(by (a))}$$

$$\implies \operatorname{rank} a + 1 \leq \operatorname{rank} A.$$

Next, Let β be an upper bound of B. We show that $\operatorname{rank} A \leq \beta$ by showing $A \subseteq V_\beta$. Indeed,

$$a \in A \implies \operatorname{rank} a + 1 \leq \beta \wedge a \in V_{\operatorname{rank} a+1} \implies a \in V_{\operatorname{rank} a+1} \subseteq V_\beta.$$

As desired. □