Chapter 7: Orderings and Ordinals

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Ordinal Numbers

Definition 7.11

An ordinal number is a transitive set well-ordered by $\in$.

Theorem 7.12

If α is an ordinal, then α^+ is also an ordinal.

Proof. • α^+ is transitive: Suppose $y \in x \in \alpha \cup \{\alpha\}$. If $y \in x \in \alpha$, then $y \in \alpha \subseteq \alpha^+$, as α is transitive. If $y \in x \in \{\alpha\}$, then $y \in x = \alpha \subseteq \alpha^+$. In both cases, we derive $y \in \alpha^+$.

• α^+ is well ordered by $\in$: Clearly, $\alpha^+ = \alpha \cup \{\alpha\}$ is linearly ordered by $\in$. Moreover, for any $\emptyset \neq A \subseteq \alpha \cup \{\alpha\}$, if $A = \{\alpha\}$, then α is the least element of $A = \{\alpha\}$. Otherwise, we have $\emptyset \neq A - \{\alpha\} \subseteq \alpha$. Then $A - \{\alpha\}$ has a least element a_0, as α is well ordered by $\in$. Since $a_0 \in \alpha$, a_0 is also the least element of A. □

Definition 7.13

We denote the successor of α by $\alpha + 1$, i.e., $\alpha + 1 = \alpha^+$.

An ordinal α is called a successor ordinal if $\alpha = \beta + 1$ for some ordinal β; otherwise, it is called a limit ordinal.

Example 7.12:

- Every nonzero natural number n^+ is a successor ordinal.
- 0 and ω are limit ordinals.
- The following are successor ordinals:
  - $\omega + 1 = \omega^+ = \omega \cup \{\omega\} = \{0, 1, 2, \ldots, \omega\}$

  - $\omega + 2 = (\omega + 1) + 1 = \omega^{++} = \omega^+ \cup \{\omega^+\} = \{0, 1, 2, \ldots, \omega, \omega^+\}$

    ⋮
  - $\omega + n + 1 = \{0, 1, 2, \ldots, \omega, \omega + 1, \ldots, \omega + n\}$

Lemma 7.14

If α is an ordinal, then any $\beta \in \alpha$ is also an ordinal.

Proof. Since any ordinal α is transitive, $\beta \in \alpha$ implies $\beta \subseteq \alpha$. Hence β is well ordered by $\in$ since α is well ordered by $\in$.

To show β transitive, suppose $x \in y \in \beta$. Since α is transitive, $y \in \alpha$, thereby $x \in \alpha$. Since α is linearly ordered by $\in$, we conclude that $x \in \beta$. □

Definition 7.15

For all ordinals α and β, define

$$\alpha < \beta \quad \text{iff} \quad \alpha \in \beta.$$

For example:

- $n < \omega$ for all $n \in \omega$.
- $\alpha < \alpha^+ = \alpha \cup \{\alpha\}$ for all ordinal α.

Lemma 7.16

For any ordinals α, β, if $\alpha \subset \beta$ then $\alpha \in \beta$.

Proof. If $\alpha \subset \beta$, then $\beta - \alpha \neq \emptyset$. Let γ be the least element of $\beta - \alpha$.

Claim: $\alpha = \gamma$.

"$\supseteq$": For any $\delta \in \gamma$, since $\gamma = \min(\beta - \alpha)$, we must have that $\delta \in \alpha$.

"$\subseteq$": For any $\delta \in \alpha$, if $\delta \notin \gamma$, then as $\delta, \gamma \in \beta$ and β is linearly ordered by $\in$, we conclude that $\gamma \underline{\in} \delta$. Since α is transitive, it follows that $\gamma \in \alpha$, contradicting the choice of γ. □

It follows from the above Lemma that for any two ordinals α, β,

$$\alpha \leq \beta \iff \alpha \subseteq \beta.$$

Theorem 7.17

Let α, β, γ be ordinals.

1. *(irreflexivity) $\alpha \notin \alpha$ for any ordinal α.*
2. *(transitivity) $\alpha \in \beta \in \gamma \Longrightarrow \alpha \in \gamma$*
3. *(trichotomy) Exactly one of the following three cases holds*

$$\alpha \in \beta, \quad \alpha = \beta, \quad \beta \in \alpha.$$

4. *Every nonempty set S of ordinals has a $<$-least element, and $\min S = \bigcap S$.*

Consequently, every set of ordinal numbers is well-ordered by $<$.

Proof. (1) Otherwise the set $\alpha \cup \{\alpha\}$ would have the infinite descending chain

$$\cdots \in \alpha \in \alpha \in \alpha,$$

contradicting that fact that the ordinal $\alpha^+ = \alpha \cup \{\alpha\}$ is well-ordered by $\in$.

(2) Follows from γ being a transitive set.

(3). If two of the three cases hold, then $\alpha \in \alpha$, contradicting item (1). It remains to check that at least one of the three cases holds.

Claim: $\alpha \cap \beta$ is an ordinal. Indeed, since α and β are transitive, $\alpha \cap \beta$ is transitive. Besides, $\alpha \cap \beta \subseteq \alpha$ is well ordered by $\in$, as α is well ordered by $\in$.

Now, if $\alpha \cap \beta = \alpha$ or β, then $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$, which by Lemma 7.15 implies that at least one of the three cases holds.

Suppose $\alpha \cap \beta \subsetneq \alpha$ and $\alpha \cap \beta \subsetneq \beta$, Then $\alpha \cap \beta \in \alpha$ and $\alpha \cap \beta \in \beta$, which lead to $\alpha \cap \beta \in \alpha \cap \beta$, contradicting item (1).

(4). Let S be a nonempty set of ordinals. We first show that $\bigcap S$ is an ordinal.

Take $\alpha \in S$. If $\bigcap S = \alpha$, then $\bigcap S$ is an ordinal. If $\bigcap S \subset \alpha$, then $\bigcap S \in \alpha$ is an ordinal by Lemma 7.16 and Lemma 7.14.

Clearly, $\bigcap S \subseteq \beta$ for all $\beta \in S$, which implies that $\bigcap S \leq \beta$ for all $\beta \in S$. To show that $\min S = \bigcap S$, it then remains to show that $\bigcap S \in S$.

Assume $\bigcap S \notin S$. Then $\bigcap S \subsetneq \beta$ for all $\beta \in S$. By Lemma 7.16, $\bigcap S \in \beta$ for all $\beta \in S$, thereby $\bigcap S \in \bigcap S$, contradicting item(1). □

Corollary 7.18

1. *Any transitive set of ordinal numbers is itself an ordinal number.*
2. *If S is a nonempty set of ordinals, then $\bigcup S$ is an ordinal.*

Proof. (1). By the preceding theorem, every set S of ordinals is well-ordered by $\in$. If S is in addition a transitive set, then S is an ordinal number.

(2). By (1), it suffices to prove that $\bigcup S$ is transitive.

For any $x \in \bigcup S$, we have that $x \in \alpha$ for some $\alpha \in S$. Since the ordinal α is transitive, $x \subseteq \alpha \subseteq \bigcup S$. Hence $\bigcup S$ is transitive. □

Remark: Slearly, $\bigcup S = \sup S$. If S contains a greatest element γ, then $\sup S = \gamma$. Otherwise, $\sup S > \gamma$ for all $\gamma \in S$.

The following are ordinals:

- $\omega + n + 1 = \{0, 1, 2, \dots, \omega, \omega + 1, \dots, \omega + n\}$
- $$\begin{aligned} \sup\{\omega + n \mid n \in \omega\} &= \bigcup\{\omega + n \mid n \in \omega\} \\ &= \{0, 1, 2, \dots, \omega, \dots, \omega + n, \dots\} \\ &= \omega + \omega = \omega \cdot 2 \end{aligned}$$
- $\omega \cdot 2 + 1 = \{0, 1, 2, \dots, \omega, \omega + 1, \omega + 2, \dots, \omega + \omega\}$

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- $$\begin{aligned} \sup\{\omega \cdot 2 + n \mid n \in \omega\} &= \{0, 1, \dots, \omega, \dots, \omega \cdot 2, \omega \cdot 2 + 1, \dots\} \\ &= \omega \cdot 2 + \omega = \omega \cdot 3 \end{aligned}$$

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- $$\begin{aligned} \sup\{\omega \cdot n \mid n \in \omega\} &= \{0, 1, \dots, \omega, \dots, \omega \cdot 2, \dots, \omega \cdot 3, \dots\} \\ &= \omega \cdot \omega = \omega^2 \end{aligned}$$
- $\sup\{\omega^n \mid n \in \omega\} = \omega^\omega$
- $\sup\{\omega^{\omega^n} \mid n \in \omega\} = \omega^{\omega^\omega}$

  ⋮

The class of all ordinals is denoted by Ord.

Theorem 7.19 (Burali-Forti Paradox)

There is no set to which every ordinal number belongs, i.e., Ord *is a proper class.*

Proof. By Lemma 7.14,

$$\beta \in \alpha \in \mathrm{Ord} \Longrightarrow \beta \in \mathrm{Ord}.$$

Thus, if Ord was a set, then it is a transitive set of ordinals. It then follows from Corollary 7.18 that the set Ord is an ordinal itself, meaning that $\mathrm{Ord} \in \mathrm{Ord}$, which is impossible. □

Corollary 7.20

The natural numbers are exactly the finite ordinal numbers.

Proof. Obviously natural numbers are finite ordinals. Conversely, if α is not a natural number, i.e., $\alpha \notin \omega$, then we must have that $\omega \in \alpha$, which implies that $\omega \subseteq \alpha$, as α is transitive. That is, α has an infinite subset, hence α is infinite. □

Lemma 7.21

Let β be an ordinal. Then $\alpha \in \beta$ iff α is an initial segment of β, i.e., $\alpha = \mathrm{seg}_\beta\,\gamma$ for some $\gamma \in \beta$.

Proof. Recall:
$$\mathrm{seg}_\beta\,\gamma = \{\delta \in \beta \mid \delta \in \gamma\}.$$

"$\Longleftarrow$": Assume $\alpha = \mathrm{seg}_\beta\,\gamma$ for some $\gamma \in \beta$. Then $\alpha \subset \beta$, which implies $\alpha \in \beta$.

"$\Longrightarrow$": Conversely, assume $\alpha \in \beta$, we prove that $\alpha = \mathrm{seg}_\beta \alpha$. Clearly, $\mathrm{seg}_\beta\,\alpha \subseteq \alpha$. Conversely, for any $\delta \in \alpha$, since $\alpha \in \beta$, we derive $\delta \in \beta$, which means $\delta \in \mathrm{seg}_\beta\,\alpha$. □

Theorem 7.22

For any ordinals α, β, if $(\alpha, \in) \cong (\beta, \in)$, then $\alpha = \beta$.

Proof. Assume $(\alpha, \in) \cong (\beta, \in)$. If $\alpha \in \beta$, then by Lemma 7.21 we derive $\alpha = \mathrm{seg}_\beta\,\alpha$. It follows that $(\mathrm{seg}_\beta\,\alpha, \in) \cong (\beta, \in)$, which is impossible. By the same argument, $\beta \notin \alpha$. Hence by trichotomy, we conclude that $\alpha = \beta$. □

Fact 7.23

Let $(A, <)$ and $(B, <')$ be two well-ordered sets.

1. *If $(A, <) \cong (\mathrm{seg}_B y, <')$ for some $y \in B$, then for any $x \in A$, we have that $(\mathrm{seg}_A x, <) \cong (\mathrm{seg}_B z, <')$ for some $z <' y$.*
2. *If $(A, <) \cong (\alpha, \in)$, then for any $x \in A$, we have that $(\mathrm{seg}_A x, <) \cong (\beta, \in)$ for some ordinal $\beta \in \alpha$.*

Proof of (2). By (1), $(\mathrm{seg}_A x, <) \cong (\mathrm{seg}_\alpha \gamma, \in)$ for some $\gamma \in \alpha$. But by Lemma 7.21, $\mathrm{seg}_\alpha \gamma \in \alpha$. □

Replacement Axioms

Let $\phi(x, y)$ be a formula not containing B such that for all x, there exists a unique y such that $\phi(x, y)$ holds. Then, for every set A, the following is a set:
$$B = \{y \mid \exists x \in A(\,\phi(x, y)\,)\}.$$

Example 7.13: If A is a set, then
$$\{\wp x \mid x \in A\} = \{y \mid \exists x \in A(y = \wp x)\}$$
is also a set.

Example 7.14: If S is a set, then
$$\{|x| : x \in S\} = \{y : \exists x \in S(y = |x|)\}$$
is also a set.

Theorem 7.24

Every well-ordered set is isomorphic to a unique ordinal.

Proof. The uniqueness of such ordinals follows from Theorem 7.22. It then suffices to prove the existence part of the theorem.
Let $(W, <)$ be a well-ordered set. Consider the set
$$A = \{x \in W \mid (\mathrm{seg}_W x, <) \cong (\alpha_x, \in) \text{ for some unique } \alpha_x\}.$$
By replacement axiom,
$$S = \{\alpha_x \mid x \in A\} = \{y \mid \exists x \in A(y = \alpha_x)\}$$
is a set. Claim: S is a transitive set, which by Corollary 7.18 will imply that the set S is an ordinal, denoted by α.

Indeed, for any $\beta \in \alpha_x \in S$, by Lemma 7.21, we have that $\beta = \mathrm{seg}_{\alpha_x}\beta$. Since $(\alpha_x, \in) \cong (\mathrm{seg}_W x, <)$, by Fact 7.23, we know that
$$(\beta, \in) = (\mathrm{seg}_{\alpha_x}\beta, \in) \cong (\mathrm{seg}_W y, <)$$
for some $y < x$, thereby $\beta = \beta_y \in S$.

Claim: Either $A = W$ or A is an initial segment of W.

Suppose $A \neq W$. Let $x_0 = \min(W - A)$. We show that

$$A = \mathrm{seg}_W x_0 = \{x \in W \mid x < x_0\}.$$

Clearly $\mathrm{seg}_W x_0 \subseteq A$. Conversely, for any $y \in A$, we have that $(\mathrm{seg}_W y, <) \cong (\alpha_y, \in)$. If $x_0 \leq y$, then $(\mathrm{seg}_W x_0, <) \cong (\beta, \in)$ for some $\beta \in \alpha_y$. But then, $x_0 \in A$, a contradiction. Hence $\mathrm{seg}_W x_0 = A$.

Define a function $f : A \to \alpha$ by taking

$$f(x) = \alpha_x.$$

We now show that f is one-to-one and order preserving.

For any $x, y \in A$ with $x < y$, since $(\mathrm{seg}_W y, <) \cong (\alpha_y, \in)$, we have that $(\mathrm{seg}_W x, <) \cong (\gamma, \in)$ for some $\gamma \in \alpha_y$. Since we also have that $(\mathrm{seg}_W x, <) \cong (\alpha_x, \in)$, by Theorem 7.22, we conclude that $\alpha_x = \gamma \in \alpha_y$.

Then $(A, <) \cong (\alpha, \in)$. If $A = \mathrm{seg}_W x_0$ for some $x_0 \in W$, then by definition, we must have that $x_0 \in A$, which is a contradiction.

Hence, $A = W$ and f is an isomorphism from W onto α, as desired. □

Transfinite Induction and Recursion

For the well-ordered set $(\omega, \in)$, we had the following induction principle:

Strong Induction Principle for ω

Let B be a subset of ω. Suppose that for every $n \in \omega$,

$$\text{if } m \in B \text{ holds for all } m \in n, \text{ then } n \in B.$$

Then $B = \omega$.

In general, for well-ordered sets, we have the following transfinite induction principle:

Transfinite Induction Principle

Let $<$ be a well ordering on a set A. Assume that B is a subset of A with the property that for every $t \in A$,

$$\mathrm{seg}_A t \subseteq B \Longrightarrow t \in B. \qquad (*)$$

Then $B = A$.

A subset B of A satisfying $(*)$ is said to be a *$<$-inductive* subset of A.

Example 7.15: Any ordinal α is well-ordered by $\in$. To show that some property $P(x)$ holds for all ordinals $\beta \in \alpha$, i.e., to show that $B = \alpha$ for

$$B = \{\beta \in \alpha \mid P(\beta)\},$$

by Transfinite Induction Principle, it suffices to show that B is $\in$-inductive, i.e., for each $\beta \in \alpha$,

$$\mathrm{seg}_\alpha \beta = \beta \subseteq B \Longrightarrow \beta \in B. \qquad (*)$$

- In fact, Transfinite Induction Principle also holds for the well-ordered class $(\mathrm{Ord}, \in)$. So to show that some property $P(x)$ holds for all ordinals, i.e., $B = \mathrm{Ord}$ for

$$B = \{\beta \in \mathrm{Ord} \mid P(\beta)\},$$

it suffices to check that $(*)$ holds for any ordinal β.

Proof. (of Transfinite Induction Principle) If $B \subsetneq A$, then $A - B$ has a least element m. Then for every $y \in A$ such that $y < m$, we must have that $y \in B$. But this means that $\mathrm{seg}\, m \subseteq B$. By the assumption, we have $m \in B$. A contradiction. □

Let α be an ordinal. Define

$$^{<\alpha}A = \{f : \beta \to A \mid \beta < \alpha \wedge f \text{ is a function}\}.$$

Transfinite Recursion Theorem (Preliminary Form)

Let A be a set, α an ordinal and $G : {}^{<\alpha}A \to A$ a function. Then there exists a unique function $F : \alpha \to A$ such that for all $\xi < \alpha$

$$F(\xi) = G(F \restriction \xi).$$

Example 7.16: Given a set A and consider the ordinal ω. Suppose we have a function $G : {}^{<\omega}A \to A$. By Transfinite Recursion Theorem, there exists a unique function $F : \omega \to A$ such that for each $n \in \omega$,

$$F(n) = G(F \restriction n).$$

That is,

$$\begin{aligned} F(0) &= G(F \restriction 0) = G(\emptyset) \\ F(1) &= G(F \restriction 1) = G(\{(0, F(0))\}) \\ F(2) &= G(F \restriction 2) = G(\{(0, F(0)), (1, F(1))\}) \\ &\vdots \end{aligned}$$

Transfinite Recursion Theorem

Let G be a function on a class. Then there exists a unique function F on Ord such that for each ordinal α,

$$F(\alpha) = G(F \restriction \alpha).$$

Example 7.17: (Ordinal Arithmetic) Fix an ordinal α. By transfinite recursion theorem, there exists a unique function $+_\alpha$ (adding α) on Ord such that

- $+_\alpha(0) = \alpha$
- $+_\alpha(\beta^+) = (+_\alpha(\beta))^+$
- $+_\alpha(\lambda) = \sup\{+_\alpha(\beta) \mid \beta < \lambda\}$ for limit ordinal λ

Hence, we can define addition $+$ on Ord recursively as

- $\alpha + 0 = \alpha$
- $\alpha + \beta^+ = (\alpha + \beta)^+$
- $\alpha + \lambda = \sup\{\alpha + \beta \mid \beta < \lambda\}$

Similarly, multiplication on Ord is defined recursively as

- $\alpha \cdot 0 = 0$
- $\alpha \cdot \beta^+ = \alpha \cdot \beta + \alpha$
- $\alpha \cdot \lambda = \sup\{\alpha \cdot \beta \mid \beta < \lambda\}$ for limit ordinal λ

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Well-Ordering Principle

For any set A, there is a well ordering on A.

Note: The usual ordering on $\mathbb{R}$ is not a well ordering, but the above principle asserts that there *exists* a well ordering on $\mathbb{R}$.

Numeration Theorem

Any set is equinumerous to some ordinal number.

Proof. By Well-Ordering Principle, any set A is well-ordered by some ordering $<$. Let α be the unique ordinal that is isomorphic to $(A, <)$. Clearly, $\alpha \approx A$. □

Definition 7.25

For any set A, define the cardinal number of A to be the least ordinal equinumerous to A, i.e., $|A| = \min\{\alpha \in \mathrm{Ord} \mid \alpha \approx A\}$.

Example 7.18: For the set $\omega + 1$, we have that $|\omega + 1| = \omega$, since $\omega \approx \omega + 1 \not\approx n$ for all $n \in \omega$. It also follows that $\aleph_0 = |\omega| = \omega$.

Example 7.19: For any sets A and B,

$$|A| = |B| \iff A \approx B.$$

Example 7.20: For any natural number n, it is easy to prove that $n \not\approx m$ for any $m \in n$. Thus natural numbers are cardinal numbers.

It follows that for a finite set A, $|A|$ is the unique natural number equinumerous to A.

To count the number of elements of a set, we count in order. Say, for the set

$$\omega = \{0, 1, 2, \dots\},$$

we can count in the order:

$$\underset{0}{\bullet} \quad \underset{1}{\bullet} \quad \underset{2}{\bullet} \quad \cdots\cdots \qquad \cong (\omega, \in)$$

Or in the order:

$$\underset{1}{\bullet} \quad \underset{2}{\bullet} \quad \underset{3}{\bullet} \quad \cdots\cdots \quad \underset{0}{\bullet} \qquad \cong (\omega + 1, \in)$$

Or:

$$\underset{2}{\bullet} \quad \underset{3}{\bullet} \quad \underset{4}{\bullet} \quad \cdots\cdots \quad \underset{0}{\bullet} \quad \underset{1}{\bullet} \qquad \cong (\omega + 2, \in)$$

We have that $\omega \approx \omega + 1 \approx \omega + 2 \approx \cdots \approx \omega + \omega \approx \omega^{\omega} \approx \dots$
We define the cardinal number of the set ω to be
$\aleph_0 = |\omega| = \min\{\omega, \omega + 1, \omega + 2, \dots\} = \omega$.