Chapter 7: Orderings and Ordinals

April 8 & 10, 2014

Lecturer: Fan Yang

Well Orderings

Definition 7.1

A relation $<$ on a set A is said to be a well ordering if $<$ is a linear ordering and every nonempty subset of A has a least element.

Example 7.1: Recall that

$$\omega = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \dots\}.$$

The ordering $\in$ on ω is a well ordering.

Example 7.2: The ordering $<$ on $\mathbb{Z}$ is not a well ordering, as, e.g., the set $\mathbb{Z}$ does not have a least element.

Example 7.3: Any linear ordering $<$ on a finite set $A = \{a_0, a_1, \dots, a_n\}$ is a well ordering, since A can be ordered as

$$a_{i_0} < a_{i_1} < \cdots < a_{i_n},$$

where $i_0, \dots, i_n \in \{0, 1, \dots, n\}$ and obviously every nonempty subset of A has a least element.

Let $<$ be a well ordering on an infinite set A. Then by definition,

- A has a least element a_0;
- $A - \{a_0\}$ has a least element a_1;
- $A - \{a_0, a_1\}$ has a least element a_2;

This way, we obtain a chain of elements of A:

$$a_0 < a_1 < a_2 < \cdots$$

- But still, if $A - \{a_0, a_1, a_2 \dots\} \neq \emptyset$, then it has a least element a_ω;
- further, if $A - \{a_0, a_1 \dots, a_\omega\} \neq \emptyset$, then it has a least element $a_{\omega+1}$;
- if $A - \{a_0, a_1, a_2 \dots, a_\omega, a_{\omega+1}\} \neq \emptyset$, then has a least element $a_{\omega+2}$;

  $\vdots$
- still, if $A - \{a_0, \dots, a_\omega, a_{\omega+1}, \dots\} \neq \emptyset$, it has a least element $a_{\omega\cdot 2}$;

  $\vdots$

Eventually, we will use up all of the elements of A and form a chain:

$$a_0 < a_1 < \cdots < a_\omega < a_{\omega+1} \cdots < a_{\omega\cdot 2} < a_{\omega\cdot 2+1} < \cdots$$

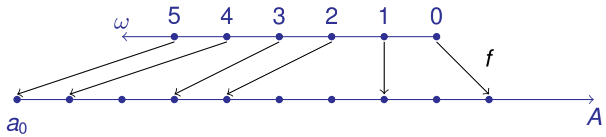
The above is an infinite ascending chain. Below is an infinite descending chain:

$$\cdots < a_2 < a_1 < a_0$$

We prove in the next theorem that a linear ordering $<$ on A is a well ordering iff A does not contain an infinite descending chain.

Theorem 7B (AC)

Let $<$ be a linear ordering on A. Then it is a well ordering iff there does not exist any function $f : \omega \to A$ with $f(n^+) < f(n)$ for every $n \in \omega$.



Proof. "$\Longrightarrow$": If there is a function $f : \omega \to A$ with $f(n^+) < f(n)$ for every $n \in \omega$, then the set $\operatorname{ran} f \subseteq A$ has no least element. This means that $<$ is not a well ordering.

"$\Longleftarrow$": Conversely, assume that $<$ is not a well ordering on A, so that some nonempty set $B \subseteq A$ lacks a least element. Define a function $f : \omega \to A$ recursively as follows:

- Pick any $b_0 \in B$ and let $f(0) = b_0$.
- For each $n^+ \in \omega$, since b_n is not a least element of B, we can choose $b_{n^+} \in B$ such that $b_{n^+} < b_n$ and define $f(n^+) = b_{n^+}$.

Note that we can make the choices by Axiom of Choice. Clearly, $f(n^+) < f(n)$ for every $n \in \omega$. □

Definition 7.2

Let $<$ be a partial ordering on a set A, and $t \in A$. The set

$$\operatorname{seg}_{(A,<)} t = \{x \in A \mid x < t\}.$$

is called the initial segment of A up to t. If the set A and the ordering $<$ are clear from the context, then we simply write $\operatorname{seg}_A t$ or $\operatorname{seg} t$.

Example 7.4: Let $(A, <)$ be any partial order. If a_0 is the least element of A, then

$$\operatorname{seg} a_0 = \{x \in A \mid x < a_0\} = \emptyset.$$

Example 7.5: Let $A = \{a_n \mid n \in \omega\}$ be well ordered by $<$ such that

$$a_0 < a_1 < a_2 < a_3 < \cdots$$

Then for each $n \in \omega$,

$$\operatorname{seg} a_{n^+} = \{x \in A \mid x < a_{n^+}\} = \{a_0, \ldots, a_n\}.$$

Example 7.6: For the well order $(\omega, \in)$, we have that for any $n \in \omega$,

$$\operatorname{seg} n = \{x \mid x \in n\} = n.$$

Isomorphisms

Definition 7.3

Let (A, R) and (B, S) be two structures. An isomorphism from (A, R) onto (B, S) is a bijection $f : A \to B$ such that for x and y in A,

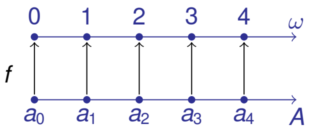
$$xRy \quad \text{iff} \quad f(x)Sf(y).$$

We say that (A, R) is isomorphic to (B, S), written $(A, R) \cong (B, S)$, if there exists an isomorphism $f : A \to B$.

Example 7.7: Consider the set $A = \{a_n \mid n \in \omega\}$ and the ordering $<$ on A defined as

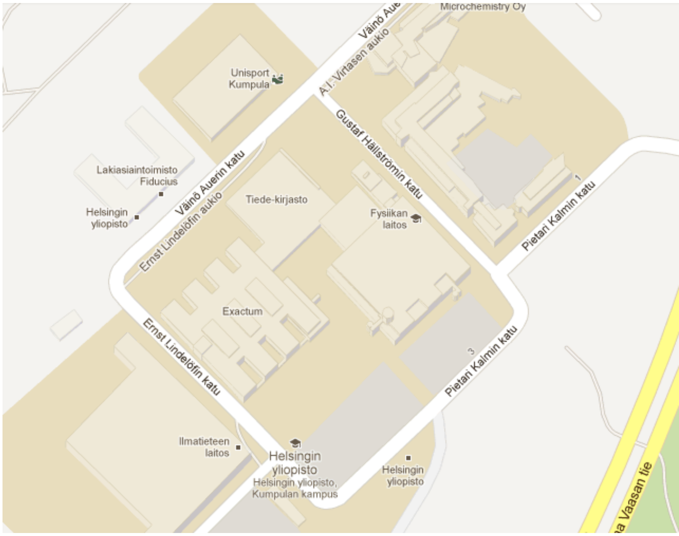
$$a_n < a_m \quad \text{iff} \quad n \in m.$$

Clearly $(A, <) \cong (\omega, \in)$, since $f : A \to \omega$ defined as $f(a_k) = k$ is an isomorphism.

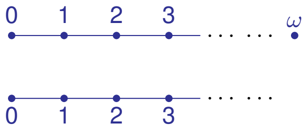


- Isomorphic structures look exactly alike.

A map of Helsinki is "isomorphic" to Helsinki



Example 7.8: Consider the set $\omega^+ = \omega \cup \{\omega\}$ and the ordering $\in$ on ω^+.



Consider also the set ω and the ordering $\in$ on ω. Clearly, $\omega \approx \omega^+$, but the two structures do not look alike.

For example, in ω, the initial segments $\operatorname{seg} x$ are always finite, but in ω^+, the initial segment $\operatorname{seg} \omega = \omega$ is infinite.

It will follow from the next lemma that $(\omega, \in) \not\cong (\omega^+, \in)$.

Lemma 7.4

Let $(A, <_A)$ and $(B, <_B)$ be partially ordered sets, and $f : A \to B$ an isomorphism. For any $a \in A$, we have that

$$f[\operatorname{seg}_A a] = \operatorname{seg}_B f(a).$$

Proof. For "$\subseteq$", we have that

$$\begin{aligned} y \in f[\operatorname{seg}_A a] &\implies y = f(x) \text{ for some } x <_A a \\ &\implies y = f(x) <_B f(a) \ (\because f \text{ is an isomorphism}) \\ &\implies y \in \operatorname{seg}_B f(a). \end{aligned}$$

To show "$\supseteq$", for any $y \in \operatorname{seg}_B f(a)$, i.e., $y <_B f(a)$, there is $x \in A$ such that $y = f(x) <_B f(a)$ since f is surjective. Hence $x <_A a$, which implies that $x \in \operatorname{seg}_A a$, thereby $y = f(x) \in f[\operatorname{seg}_A a]$. $\square$

> Example 7.9: Prove that $(\omega^+, \in) \not\cong (\omega, \in)$, where $\omega^+ = \omega \cup \{\omega\}$.

Proof. Suppose $f : \omega^+ \to \omega$ is an isomorphism. From Lemma 7.4, it follows that

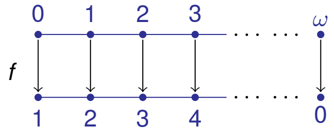
$$f[\mathrm{seg}_{\omega^+}\omega] = \mathrm{seg}_\omega f(\omega),$$

which implies that

$$\{f(n) \mid n \in \omega\} = \{m \mid m \in f(\omega)\} = f(\omega) \in \omega.$$

But since f is one-to-one, the set $\{f(n) \mid n \in \omega\}$ is infinite. This leads to a contradiction. □

Example 7.10: Consider again the set $\omega^+ = \omega \cup \{\omega\}$ and the ordering $\in$ on ω^+.



Consider the set ω and the ordering $\prec$ on ω defined as

$$\prec = \{(n, m) \mid 0 \in n \in m\} \cup \{(n, 0) \mid 0 \in n\}.$$

We have that $(\omega^+, \in) \cong (\omega, \prec)$, since the function $f : \omega^+ \to \omega$ defined below is an isomorphism:

$$f(x) = \begin{cases} x^+, & \text{if } x \in \omega; \\ 0, & \text{if } x = \omega. \end{cases}$$

Theorem 7E

Let (A, R), (B, S), and (C, T) be any structures. Then

1. $(A, R) \cong (A, R)$;
2. $(A, R) \cong (B, S) \Longrightarrow (B, S) \cong (A, R)$;
3. $(A, R) \cong (B, S) \cong (C, T) \Longrightarrow (A, R) \cong (C, T)$.

Proof. (1) Let id_A be the identity map on A. Then for any $x, y \in A$,

$$xRy \iff \mathrm{id}_A(x) R \, \mathrm{id}_A(y).$$

Hence $(A, R) \cong (A, R)$.

(2) Assume $(A, R) \cong (B, S)$. Then, there is a bijection $f : A \to B$ such that xRy iff $f(x)Sf(y)$. For any $x', y' \in B$, there exist $x, y \in A$ such that $f(x) = x'$ and $f(y) = y'$. Hence

$$x'Sy' \iff f(x)Sf(y) \iff xRy \iff f^{-1}(x')Rf^{-1}(y'),$$

which means $(B, S) \cong (A, R)$;

(3) Assume $(A, R) \cong (B, S) \cong (C, T)$. Then, there are bijections $f : A \to B$ and $g : B \to C$ such that

$$xRy \text{ iff } f(x)Sf(y) \quad \text{and} \quad x'Sy' \text{ iff } g(x')Tg(y').$$

Hence $g \circ f$ is a bijection, and $xRy \iff f(x)Sf(y) \iff g(f(x))Tg(f(y))$, which means $(A, R) \cong (C, T)$. □

Lemma 7F

Let $(A, <_A)$ and $(B, <_B)$ be two structures. Assume that $f : A \to B$ is a one-to-one function such that for any $x, y \in A$,

$$x <_A y \quad \text{iff} \quad f(x) <_B f(y). \qquad (*)$$

1. *If $<_B$ is a partial ordering on B, then $<_A$ is a partial ordering on A.*
2. *If $<_B$ is a linear ordering on B, then $<_A$ is a linear ordering on A.*
3. *If $<_B$ is a well ordering on B, then $<_A$ is a well ordering on A.*

Proof. We only prove Item 3. It suffices to show that any nonempty subset S of A has a least element. Since $f[S]$ is a nonempty subset of B, and since $<_B$ is a well ordering on B, there is $m' \in f[S]$ such that

$$m' \leq_B f(x) \text{ for all } x \in S.$$

As f is one-to-one, there exists a unique $m \in S$ such that $f(m) = m'$. From $(*)$, it follows that

$$m \leq x \text{ for all } x \in S.$$

Hence m is the least element of S. □

Theorem 7G

Assume that structures $(A, <_A)$ and $(B, <_B)$ are isomorphic. If one is a partially (or linearly or well) ordered structure, so also is the other.

Proof. It follows from Lemma 7F, since if $f : A \to B$ is an isomorphism, then both f and f^{-1} are one-to-one and satisfy $(*)$. $\square$

Lemma 7.5

Let $(A, <)$ be a well-ordered set. If $f : A \to A$ is a function satisfying

$$x < y \Longrightarrow f(x) < f(y) \qquad (*)$$

for all $x, y \in A$, then $f(x) \geq x$ for all $x \in A$.

Proof. Assume that there exists $a \in A$ such that $f(a) < a$. Then

$$B = \{x \in A : f(x) < x\}$$

is a nonempty subset of A. Let x_0 be the least element of B. Hence $f(x_0) < x_0$. It follows that

$$f(f(x_0)) < f(x_0),$$

thereby $f(x_0) \in B$. As x_0 is the least element of B, $x_0 \leq f(x_0)$; a contradiction. $\square$

Corollary 7.6

Let $(A, <)$ be a well-ordered set. If $f : A \to A$ is an isomorphism, then f is the identity function on A.

Proof. If $f : A \to A$ is an isomorphism, then $f^{-1} : A \to A$ is also an isomorphism. By Lemma 7.5, we have that

$$\forall x \in A \left(f(x) \geq x \wedge f^{-1}(x) \geq x \right)$$
$$\Longrightarrow \forall x \in A \left(f(x) \geq x \wedge f(f^{-1}(x)) \geq f(x) \right)$$
$$\Longrightarrow \forall x \in A (f(x) = x),$$

thus f is the identity function on A. $\square$

Corollary 7.7

Let $(A, <)$ and $(B, <')$ be well-ordered sets. If $(A, <) \cong (B, <')$, then the isomorphism from A onto B is unique.

Proof. Let $f, g : A \to B$ be isomorphisms. Then $f^{-1} \circ g : A \to A$ is an isomorphism. By Corollary 7.6, $f^{-1} \circ g = \mathrm{id}_A$, thereby $g = f$. $\square$

Corollary 7.8

No well-ordered set is isomorphic to an initial segment of itself.

Proof. Let $(A, <)$ be a well-ordered set. If $f : A \to \operatorname{seg} t$ is an isomorphism, for some $t \in A$, then $f(t) \in \operatorname{seg} t$, i.e., $f(t) < t$, contrary to Lemma 7.5. $\square$

From Corollary 7.8 and Lemma 7.4, we know the following fact:

Fact 7.9

1. *If f is as in Lemma 7.4, then $f \upharpoonright \mathrm{seg}_A a$ is an isomorphism from $\mathrm{seg}_A a$ onto $\mathrm{seg}_B f(a)$, thereby $(\mathrm{seg}_A a, <_A) \cong (\mathrm{seg}_B f(a), <_B)$.*
2. *Let $(A, <)$ be a well-ordered set. If $a, b \in A$ and $(\mathrm{seg}\, a, <) \cong (\mathrm{seg}\, b, <)$, then $a = b$.*

Theorem 7.10

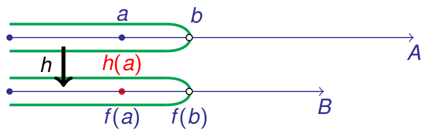
If $(A, <)$ and $(B, <')$ are well-ordered sets, then exactly one of the following three cases holds:

1. *$(A, <) \cong (B, <')$;*
2. *$(A, <)$ is isomorphic to an initial segment of $(B, <')$;*
3. *$(B, <')$ is isomorphic to an initial segment of $(A, <)$.*

Proof. In view of Corollary 7.8, the three cases are mutually exclusive. We now proceed to prove that at least one of the three cases holds.

Construct a function as follows:

$$f = \{(x, y) \in A \times B \mid (\mathrm{seg}_A x, <) \cong (\mathrm{seg}_B y, <')\}.$$



Given any $a, b \in \mathrm{dom}\, f$ with $a < b$, we show that $f(a) < f(b)$, which will imply that f is one-to-one and ordering-preserving. By assumption, there exists an isomorphism

$$h : \mathrm{seg}_A b \to \mathrm{seg}_B f(b).$$

Then $h \upharpoonright \mathrm{seg}_A a$ is an isomorphism from $\mathrm{seg}_A a$ onto $\mathrm{seg}_B h(a)$. Thus $(\mathrm{seg}_B h(a), <') \cong (\mathrm{seg}_A a, <) \cong (\mathrm{seg}_B f(a), <')$, implying $f(a) = h(a) < f(b)$.

Claim:
- If $\mathrm{dom}\, f \neq A$, then $\mathrm{dom}\, f = \mathrm{seg}_A x_0$, where $x_0 = \min(A - \mathrm{dom}\, f)$.
- If $\mathrm{ran}\, f \neq B$, then $\mathrm{ran}\, f = \mathrm{seg}_B y_0$, where $y_0 = \min(B - \mathrm{ran}\, f)$.

By the above claim, if $\mathrm{dom}\, f = A$ or $\mathrm{ran}\, f = B$, then one of the three cases holds.

To show that $\mathrm{dom}\, f = A$ or $\mathrm{ran}\, f = B$, assume $\mathrm{dom}\, f \neq A$ and $\mathrm{ran}\, f \neq B$. Since $(\mathrm{dom}\, f, <) \cong (\mathrm{ran}\, f, <')$, we have that $(\mathrm{seg}_A x_0, <) \cong (\mathrm{seg}_B y_0, <')$, which means $x_0 \in \mathrm{dom}\, f$ by the definition of f, contradicting $x_0 \in A - \mathrm{dom}\, f$.

Proof of Claim: We only prove the first item. "$\supseteq$": For any x, we have that

$$x \in \mathrm{seg}_A x_0 \Longrightarrow x < x_0 \Longrightarrow x \notin A - \mathrm{dom}\, f \Longrightarrow x \in \mathrm{dom}\, f.$$

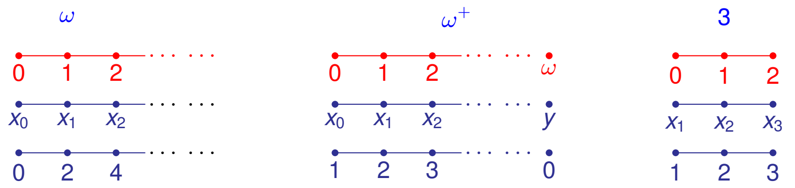
"$\subseteq$": Given any $x \in \mathrm{dom}\, f$, assume $x \notin \mathrm{seg}_A x_0$, i.e., $x_0 \leq x$. By assumption, there are $y \in B$ and an isomorphism $h : \mathrm{seg}_A x \to \mathrm{seg}_B y$. Thus $(\mathrm{seg}_A x_0, <) \cong (\mathrm{seg}_B h(x_0), <')$, thereby $x_0 \in \mathrm{dom}\, f$, contradicting $x_0 \in A - \mathrm{dom}\, f$. $\dashv$

This completes the proof. $\square$

Ordinal Numbers

If two well-ordered sets $(A, <)$ and $(B, <')$ are isomorphic, i.e., $(A, <) \cong (B, <')$, then we say that they have the same *order type*.

In each of the following groups of well-ordered sets, the three well-ordered sets have the same order type:



For each group of sets of same order type, we want to assign a set that represents the order type. Such sets are called *ordinal numbers*, namely, ordinal numbers are representatives for all well-ordered sets.

- The natural number $3 = \{0, 1, 2\} = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$ is well-ordered by $\in$ and is a transitive set.
- The set $\omega = \{0, 1, 2, \dots\}$ is transitive set well-ordered by $\in$.

Recall: A set X is transitive iff $x \in y \in X \implies x \in X$.

Definition 7.11

An ordinal number is a transitive set well-ordered by $\in$.

Example 7.11:

- Every natural number n is an ordinal number. In particular, $0 = \emptyset$ is an ordinal number.
- ω is an ordinal number.

Ordinals are usually denoted by lowercase Greek letters $\alpha, \beta, \gamma, \dots$.