Elements of Set Theory

Chapter 6: Cardinal Numbers and The Axiom of Choice

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Ordering Cardinal Numbers

Cantor-Schröder-Bernstein Theorem

(a) *If $A \preccurlyeq B$ and $B \preccurlyeq A$, then $A \approx B$.*

(b) *For any cardinals κ and λ, $\kappa \leq \lambda \leq \kappa \Longrightarrow \kappa = \lambda$.*

Proof. (b) is an immediate consequence of (a), so it suffices to prove (a).

But (a) follows from the lemma below:

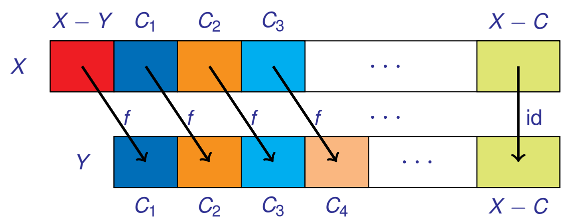
Lemma 6.11

If $X \preccurlyeq Y$ and $Y \subseteq X$, then $X \approx Y$.

Indeed, suppose $A \preccurlyeq B$ and $B \preccurlyeq A$. Then there exists $B' \subseteq A$ such that $B' \approx B$. Hence we have that $A \preccurlyeq B'$ and $B' \subseteq A$. By Lemma 6.11, $A \approx B' \approx B$, as desired.

Proof of Lemma 6.11. Let $f : X \to Y$ be a one-to-one function, and $Y \subseteq X$. For each $n \in \omega$, define C_n recursively as follows:

$$C_0 = X - Y, \qquad C_{n^+} = f[C_n].$$



Put $C = \bigcup_{n\in\omega} C_n$. Claim: The function $h : X \to Y$ defined below is a bijection:

$$h(x) = \begin{cases} f(x), & x \in C; \\ x, & x \in X - C. \end{cases}$$

- h is surjective. Indeed, given any $y \in Y \subseteq X$, either $y \in X - C$ or $y \in C$.

If $y \in X - C$, then $y = h(y)$.

If $y \in C$, then as $y \notin X - Y = C_0$, we have that $y \in C_{n^+} = f[C_n]$ for some $n \in \omega$. Hence there is $x \in C_n$ such that $y = f(x) = h(x)$.

- To see that h is injective, take any $x, x' \in X$ such that $x \neq x'$.

If $x, x' \in C$, then $h(x) = f(x) \neq f(x') = h(x')$, as f is injective.

If $x, x' \in X - C$, then $h(x) = x \neq x' = h(x')$.

If $x \in C$ and $x' \notin C$, then $x \in C_n$ for some $n \in \omega$ and $h(x) = f(x) \neq x' = h(x')$, as otherwise $x' \in f[C_n] = C_{n^+} \subseteq C$. ⊣

□

Example 6.28: $\mathbb{R} \approx {}^{\omega}2 \approx \wp\omega$.

Proof. We have proved ${}^{\omega}2 \approx \wp\omega$, so it remains to check $\mathbb{R} \approx {}^{\omega}2$, which is reduced to proving $[0,1] \approx {}^{\omega}2$, as $\mathbb{R} \approx [0,1]$, By Cantor-Schröder-Bernstein Theorem, it suffices to prove that $[0,1] \preccurlyeq {}^{\omega}2$ and ${}^{\omega}2 \preccurlyeq [0,1]$.

To show $[0,1] \preccurlyeq {}^{\omega}2$, it suffices to construct a one-to-one function $H : [0,1] \to {}^{\omega}2$. Now, represent every number in $[0,1]$ by its binary expansion.[1] A typical such binary expansion is

$$0.1100010\ldots$$

H maps the above number to the function $f : \omega \to 2$ whose successive values are

$$1, \quad 1, \quad 0, \quad 0, \quad 0, \quad 1, \quad 0, \ldots$$

In general, for any real number $z \in [0,1]$, let $H(z) : \omega \to 2$ be the function whose value at n equals the $(n+1)$st bit in the binary expansion of z. Clearly, H is one-to-one.

To show ${}^{\omega}2 \preccurlyeq [0,1]$, we define a function $G : {}^{\omega}2 \to [0,1]$ by taking

$$G(f) = 0.f(0)f(1)f(2)\ldots$$

Clearly, G is a one-to-one function. □

[1] Note that $0.1 = 0.1000\cdots = 0.0111\cdots = \frac{1}{2}$. For definiteness, always select the nonterminating binary expansion.

As a consequence of the above example, we have that

- $|\mathbb{R}| = 2^{\aleph_0}$
- $|\mathbb{R} \times \mathbb{R}| = 2^{\aleph_0} \cdot 2^{\aleph_0} = 2^{\aleph_0 + \aleph_0} = 2^{\aleph_0}$
- So the line $\mathbb{R}$ is equinumerous to the plane $\mathbb{R} \times \mathbb{R}$.

Theorem 6L

Let κ, λ, μ be cardinals.

(a) $\kappa \leq \lambda \Longrightarrow \kappa + \mu \leq \lambda + \mu$

(b) $\kappa \leq \lambda \Longrightarrow \kappa \cdot \mu \leq \lambda \cdot \mu$

(c) $\kappa \leq \lambda \Longrightarrow \kappa^{\mu} \leq \lambda^{\mu}$

(d) $0 < \kappa \leq \lambda \Longrightarrow \mu^{\kappa} \leq \mu^{\lambda}$

Proof. Let K, L, M be sets such that $|K| = \kappa$, $|L| = \lambda$ and $|M| = \mu$, as well as $K \subseteq L$ and $L \cap M = \emptyset$.
(a), (b) and (c) are immediate, since

$$K \cup M \subseteq L \cup M, \quad K \times M \subseteq L \times M \quad \text{and} \quad {}^{M}K \subseteq {}^{M}L.$$

For (d), if $\mu = 0$, then as $\kappa > 0$, we have $\mu^{\kappa} = 0^{\kappa} = 0 \leq \mu^{\lambda}$. Now assume $\mu \neq 0$, i.e., $M \neq \emptyset$. We construct a one-to-one function $G : {}^{K}M \to {}^{L}M$. Pick $a \in M$. For any $f \in {}^{K}M$, define $G(f) : L \to M$ as

$$G(f)(x) = \begin{cases} f(x) & \text{if } x \in K, \\ a & \text{if } x \in L \setminus K. \end{cases}$$

G is one-to-one, as $G(f_1) = G(f_2) \Longrightarrow \forall x \in K[f_1(x) = f_2(x)] \Longrightarrow f_1 = f_2$.
This completes the proof. □

Example 6.29: Compute $\aleph_0 \cdot 2^{\aleph_0}$.

Solution. Since

$$2^{\aleph_0} = 1 \cdot 2^{\aleph_0} \leq \aleph_0 \cdot 2^{\aleph_0} \leq 2^{\aleph_0} \cdot 2^{\aleph_0} = 2^{\aleph_0},$$

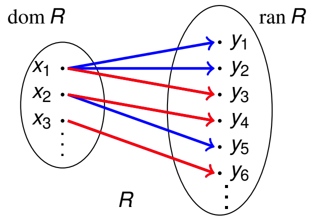
by Cantor-Schröder-Bernstein Theorem, $\aleph_0 \cdot 2^{\aleph_0} = 2^{\aleph_0}$. □

The above example shows that $\kappa < \lambda$ does not necessarily imply $\kappa \cdot \mu < \lambda \cdot \mu$ even if $\mu > 0$.

Axiom of Choice

Axiom of Choice (1st form)

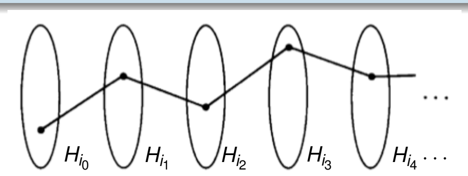
For any relation R, there is a function $F \subseteq R$ with $\operatorname{dom} F = \operatorname{dom} R$.



Example 6.30: $B \preccurlyeq A$ iff there is a surjective function $g : A \to B$.

Axiom of Choice (2nd form)

For any set I and any function H with $I \subseteq \operatorname{dom} H$, if $H_i \neq \emptyset$ for all $i \in I$, then $\prod_{i \in I} H_i \neq \emptyset$.



Properties of $\leq$ on cardinals:

1. $\kappa \leq \kappa$.
2. $\kappa \leq \lambda \leq \mu \Longrightarrow \kappa \leq \mu$.
3. $\kappa \leq \lambda \leq \kappa \Longrightarrow \kappa = \lambda$. (Cantor-Schröder-Bernstein Theorem)
4. $\kappa \leq \lambda < \mu \Longrightarrow \kappa < \mu$ and $\kappa < \lambda \leq \mu \Longrightarrow \kappa < \mu$ (follows from C-S-B Thm)
5. Either $\kappa \leq \lambda$ or $\lambda \leq \kappa$. (An immediate consequence of the 5th form of Axiom of Choice)

Axiom of Choice (5th form)

For any sets C and D, either $C \preccurlyeq D$ or $D \preccurlyeq C$.

Hence, the ordering $<$ on cardinals is:

- a partial ordering, as it is
  - irreflexive: $\kappa \not< \kappa$
  - and transitive: $\kappa < \lambda < \mu \Longrightarrow \kappa < \mu$
- a linear ordering, as it is
  - transitive
  - and satisfies trichotomy: exactly one of the following holds

$$\kappa < \lambda, \quad \kappa = \lambda, \quad \lambda < \kappa.$$

Definition 6.12

- *A* *structure* (A, R) *is a pair consisting of a set* A *and a binary relation* R *on* A *(i.e.,* $R \subseteq A \times A$*).*
- *If* $(\mathscr{A}, <)$ *is a structure such that* $<$ *is a partial ordering, then we say that* $\mathscr{A}$ *is* *partially ordered by* $<$*, or* $(\mathscr{A}, <)$ *is a* *partial order*.
- *A* *chain* $\mathscr{B}$ *of a partial order* $(\mathscr{A}, <)$ *is a subset of* $\mathscr{A}$ *such that for any* $C, D \in \mathscr{B}$*, either* $C \leq D$ *or* $D \leq C$*.*

Axiom of Choice (6th form, Zorn's Lemma)

Let $(\mathscr{A}, <)$ *be a partial order. If every chain of* $\mathscr{A}$ *has an upper bound in* $\mathscr{A}$*, then* $\mathscr{A}$ *has a maximal element.*

Observing that $\subset$ is a partial ordering on any set $\mathscr{A}$, the following is a special case of Zorn's Lemma. (In fact, the two forms of Zorn's Lemma are equivalent.)

> Given any set $\mathscr{A}$, if $\bigcup \mathscr{B} \in \mathscr{A}$ for every chain $\mathscr{B}$ of $(\mathscr{A}, \subset)$, then $\mathscr{A}$ contains an element M such that $M \not\subset a$ for all $a \in \mathscr{A}$ with $M \neq a$.

Proof. Let R be a relation. To apply Zorn's Lemma, consider the set

$$\mathscr{A} = \{f \subseteq R \mid f \text{ is a function}\}.$$

and the partial ordering $\subset$ on $\mathscr{A}$. Claim: for every chain $\mathscr{B}$ of $(\mathscr{A}, \subset)$, we have $\bigcup \mathscr{B} \in \mathscr{A}$.

Indeed, since $f \subseteq R$ for all $f \in \mathscr{B} \subseteq \mathscr{A}$, we have $\bigcup \mathscr{B} \subseteq R$. To see that $\bigcup \mathscr{B}$ is a function, it suffices to show that $y = z$ if $(x, y), (x, z) \in \bigcup \mathscr{B}$.

For any $(x, y), (x, z) \in \bigcup \mathscr{B}$, there are functions $g, h \in \mathscr{B} \subseteq \mathscr{A}$ such that

$$(x, y) \in g \in \mathscr{B} \quad \text{and} \quad (x, z) \in h \in \mathscr{B}.$$

As $\mathscr{B}$ is a chain, either $g \subseteq h$ or $h \subseteq g$. In both cases, (x, y) and (x, z) belong to a single function, thereby $y = z$. $\dashv$

By Zorn's Lemma, $\mathscr{A}$ contains a maximal function F. If $\mathrm{dom} F \subsetneq \mathrm{dom} R$, i.e., if there exists $x \in \mathrm{dom}\, R - \mathrm{dom}\, F$, then there is some y with xRy, thereby

$$F' = F \cup \{(x, y)\} \in \mathscr{A},$$

which contradicts the maximality of F. Hence $\mathrm{dom} F = \mathrm{dom} R$. □

Theorem 6M

Zorn's Lemma $\Longrightarrow$ *5th form of Axiom of Choice.*

Proof. Let C and D be two sets. To utilize Zorn's Lemma, consider

$$\mathscr{A} = \{f \mid f \text{ is a one-to-one function } \wedge \mathrm{dom}\, f \subseteq C \wedge \mathrm{ran}\, f \subseteq D\}.$$

Claim: every chain $\mathscr{B}$ of $(\mathscr{A}, \subset)$ has $\bigcup \mathscr{B}$ as an upper bound in $\mathscr{A}$, i.e., $\bigcup \mathscr{B} \in \mathscr{A}$.

By an argument similar to the previous one, we can show that $\bigcup \mathscr{B}$ is a one-to-one function. Next, consider any $(x, y) \in \bigcup \mathscr{B}$. Then $(x, y) \in f \in \mathscr{B} \subseteq \mathscr{A}$ for some f, which implies that $x \in C$ and $y \in D$. Hence $\mathrm{dom} \bigcup \mathscr{B} \subseteq C$ and $\mathrm{ran} \bigcup \mathscr{B} \subseteq D$. In conclusion, $\bigcup \mathscr{B} \in \mathscr{A}$. $\dashv$

Now, by Zorn's Lemma, $\mathscr{A}$ contains a maximal element $\hat{f}$.

Claim: Either $\mathrm{dom}\, \hat{f} = C$ (which implies $C \preccurlyeq D$), or $\mathrm{ran}\, \hat{f} = D$ (which implies $D \preccurlyeq C$, for then $\hat{f}^{-1} : D \to C$ is a one-to-one function).

Assume otherwise. Then there exist $c \in C - \mathrm{dom}\, \hat{f}$ and $d \in D - \mathrm{ran}\, \hat{f}$, and the one-to-one function

$$f' = \hat{f} \cup \{(c, d)\}$$

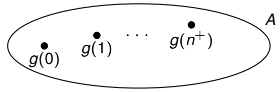
is in $\mathscr{A}$, contradicting the maximality of $\hat{f}$. □

Theorem 6N

(a) *For any infinite set* A*, we have* $\omega \preccurlyeq A$*.*

(b) $\aleph_0 \leq \kappa$ *for any infinite cardinal* κ*.*

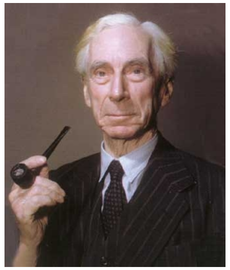
Proof. (idea) It suffices to prove (a). We attempt to define a one-to-one function $g : \omega \to A$ by recursion:



$$\begin{aligned} g(0) &= \text{some element of } A \\ g(1) &= \text{some element of } A - \{g(0)\} \\ &\vdots \\ g(n^+) &= \text{some element of } A - \{g(0), \ldots, g(n)\}. \end{aligned}$$

Since A is infinite, the set $A - \{g(0), \ldots, g(n)\}$ is nonempty for all $n \in \omega$.

But the phrase "some element" is unclear. Unless we specify *which* element, the above cannot possibly be construed as defining g. For this purpose, we need (the 3rd form of) Axiom of Choice. □



Bertrand Russell
(1872-1970)

Given a set of infinite many pairs of shoes. ZF axioms allow you to say "form a new set by picking the left shoe from each pair".

But if they are infinitely many pairs of indistinguishable socks, then there is no explicit way for you to say "choose one sock from each pair". Axiom of Choice is required here.

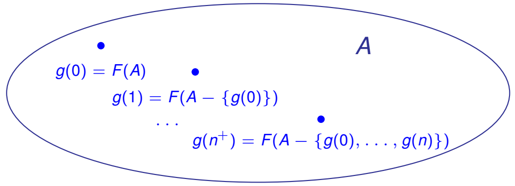
Axiom of Choice (3rd form)

For any set A, there is a function F (called a "choice function" for A) such that the domain of F is the set of all nonempty subsets of A, and $F(B) \in B$ for every nonempty $B \subseteq A$.

Theorem 6N (AC)

(a) *For any infinite set A, we have $\omega \preccurlyeq A$.*
(b) *$\aleph_0 \leq \kappa$ for any infinite cardinal κ.*

Proof. It suffices to prove (a). By the 3rd form of Axiom of Choice, there is a choice function $F : \wp A \setminus \{\emptyset\} \to A$ such that $F(B) \in B$ for all $\emptyset \neq B \subseteq A$. We then attempt to define a one-to-one function $g : \omega \to A$ by recursion:



But to apply recursion theorem, we need to define an auxiliary function $h : \omega \to \wp A$ recursively as follows:

$$h(0) = \emptyset$$
$$h(n^+) = h(n) \cup \{F(A - h(n))\}.$$

We now define $g : \omega \to A$ from h as follows:

$$g(n) = F(A - h(n)).$$

Claim: g is one-to-one.
Let m, n be two distinct natural numbers. W.l.o.g., assume $m \in n$. Then $m^+ \in n$ and

$$g(m) \in h(m^+) \subseteq h(n).$$

Since F is a choice function, we have that

$$g(n) = F(A - h(n)) \in A - h(n),$$

implying $g(n) \notin h(n)$. This shows that $g(m) \neq g(n)$, as required. □

Theorem 6M

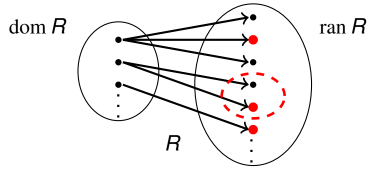
1st form of AC $\iff$ 3rd form of AC.

Proof. "1st $\Longrightarrow$ 3rd": Let A be a set. Consider the following relation

$$R = \{\{B\} \times B \mid \emptyset \neq B \subseteq A\}.$$

By 1st form of AC, there is a function $F \subseteq R$ with $\operatorname{dom} F = \operatorname{dom} R = \wp A \setminus \{\emptyset\}$. Clearly, for each nonempty $B \subseteq A$, we have $F(B) \in B$.

"3rd $\Longrightarrow$ 1st": Let R be a relation. By the 3rd form of AC, there is a choice function G for $\operatorname{ran} R$ such that $G(B) \in B$ for all nonempty $B \subseteq \operatorname{ran} R$.

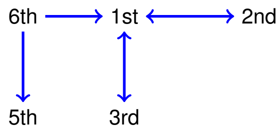


Define a function F with $\operatorname{dom} F = \operatorname{dom} R$ by putting

$$F(x) = G(\{y \mid xRy\}).$$

Then $F(x) \in \{y \mid xRy\}$, thereby $xRF(x)$. Hence $F \subseteq R$. □

What we have proved so far:



Some immediate consequences of Theorem 6N:

1. $|A| = \aleph_0$, for any infinite subset A of ω.
2. $\kappa < \aleph_0$ iff κ is finite.
3. Any subset B of a finite set A is finite, since by Cantor-Schröder-Bernstein Theorem,

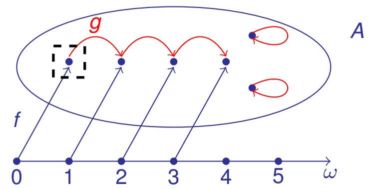
$$|B| \leq |A| < \aleph_0 \Longrightarrow |B| < \aleph_0.$$

Corollary 6P

A set is infinite iff it is equinumerous to a proper subset of itself.

Proof. The direction "$\Longleftarrow$" follows from Corollary 6D. For "$\Longrightarrow$", assume A is infinite and $f : \omega \to A$ is a one-to-one function. Define a function $g : A \to A$ by taking

$$g(f(n)) = f(n^+), \qquad \text{for } n \in \omega;$$
$$g(x) = x, \qquad \text{for } x \notin \operatorname{ran} f.$$



Clearly, g is a one-to-one function from A onto $A - \{f(0)\}$. Hence $A \approx A - \{f(0)\}$. □

Countable Sets

Definition 6.13

A set A is countable iff $|A| \leq \aleph_0$.

That is, a set A is countable iff there is a one-to-one function $f : A \to \omega$,
iff there is a surjective function $g : \omega \to A$,
iff A is finite or $|A| = \aleph_0$

Example 6.31:

- The set ω, $\mathbb{Z}$, $\mathbb{Q}$ are infinite countable sets. But $\mathbb{R}$ is uncountable, as $|\mathbb{R}| = 2^{\aleph_0} > \aleph_0$.
- Let A and B be two countable sets.
  - For any $C \subseteq A$, we have
    $$|C| \leq |A| = \aleph_0,$$
    thus C is countable.
  - $A \cup B$ and $A \times B$ are countable, since
    $$|A \cup B| = |A \cup (B \setminus A)| = |A| + |B \setminus A| \leq \aleph_0 + \aleph_0 = \aleph_0,$$
    $$|A \times B| = \aleph_0 \cdot \aleph_0 = \aleph_0.$$
- If A is infinite, then $\wp A$ is not countable, since
  $$|\wp A| = 2^{|A|} \geq 2^{\aleph_0} > \aleph_0.$$

Theorem 6Q (AC)

A countable union of countable sets is countable. That is, if $\mathscr{A}$ is a countable set such that every member of $\mathscr{A}$ is a countable set, then $\bigcup\mathscr{A}$ is countable.

Proof. If $\mathscr{A}=\emptyset$, then by definition, $\bigcup\mathscr{A}=\bigcup\emptyset=\emptyset$. Now, assume $\mathscr{A}\neq\emptyset$. W.l.o.g., we may assume that $\emptyset\notin\mathscr{A}$, for $\bigcup\mathscr{A}=\bigcup(\mathscr{A}-\{\emptyset\})$.

To show that $|\bigcup\mathscr{A}|\leq\aleph_0=|\omega\times\omega|$, it suffices to construct a surjective function $f:\omega\times\omega\to\bigcup\mathscr{A}$.

Since $|\mathscr{A}|\leq\aleph_0$, there is a surjective function $G:\omega\to\mathscr{A}$. For each $m\in\omega$, $G(m)$ is an element of $\mathscr{A}$ and $|G(m)|\leq\aleph_0$ by assumption. By Axiom of Choice, we can choose a surjection $F_m:\omega\to G(m)$ for each $m\in\omega$. Now, define $f:\omega\times\omega\to\bigcup\mathscr{A}$ as

$$f(m,n)=F_m(n).$$

It remains to check that f is surjective. For any $a\in\bigcup\mathscr{A}$, there exists $A\in\mathscr{A}$ such that $a\in A$. Since $G:\omega\to\mathscr{A}$ is surjective, $G(m)=A$ for some $m\in\omega$. Then, as $F_m:\omega\to G(m)$ is surjective, $a\in A=G(m)$ has a pre-image $n\in\omega$ under F_m, i.e., $a=F_m(n)=f(m,n)$. □

Miscellaneous

Arithmetic of Infinite Cardinals

Lemma 6R

For any infinite cardinal κ, we have $\kappa\cdot\kappa=\kappa$.

Proof. We will not go through the proof in this intermediate level course. □

Absorption Law of Cardinal Arithmetic

Let κ and λ be cardinals such that $\max\{\kappa,\lambda\}\geq\aleph_0$ *and* $\min\{\kappa,\lambda\}\neq 0$. *Then*

$$\kappa+\lambda=\kappa\cdot\lambda=\max\{\kappa,\lambda\}.$$

Proof. W.l.og., we may assume that $1\leq\lambda\leq\kappa$. By Lemma 6R, we have that

$$\kappa\leq\kappa+\lambda\leq\kappa+\kappa=2\cdot\kappa\leq\kappa\cdot\kappa=\kappa.$$

and

$$\kappa\leq\kappa\cdot\lambda\leq\kappa\cdot\kappa=\kappa.$$

Hence $\kappa+\lambda=\kappa=\kappa\cdot\lambda$ by Cantor-Schröder-Bernstein Theorem. □

Arithmetic of Infinite Cardinals

Example 6.32: For any infinite cardinal κ, we have $\kappa^\kappa=2^\kappa$, since

$$\kappa^\kappa\leq(2^\kappa)^\kappa=2^{\kappa\cdot\kappa}=2^\kappa\leq\kappa^\kappa.$$

Example 6.33: The total number of functions from $\mathbb{R}$ to $\mathbb{R}$ is

$$|{}^{\mathbb{R}}\mathbb{R}|=(2^{\aleph_0})^{2^{\aleph_0}}=2^{\aleph_0\cdot 2^{\aleph_0}}=2^{2^{\aleph_0}}.$$

Continuum Hypothesis

Infinite countable sets have cardinality $\aleph_0$.

Every uncountable set we examined so far has cardinality $2^{\aleph_0}$.

This raises the question: are there any sets with cardinality between $\aleph_0$ and $2^{\aleph_0}$?

Continuum Hypothesis (CH)

There is no cardinal κ with $\aleph_0 < \kappa < 2^{\aleph_0}$.

On the basis of ZFC axioms (which we assume to be consistent):

- Kurt Gödel (1939): CH cannot be *disproved*.
- Paul Cohen (1963): CH cannot be *proved* either. (By a technique called *forcing*.)

So, CH is *independent* of ZFC axioms. This is a topic of the advanced level course "*Axiomatic Set Theory*".