DEDEKIND ETA FUNCTION

$$\boxed{\eta(\tau) = e^{i\frac{\pi}{12}\tau} \cdot \prod_{k=1}^{\infty} \left(1 - e^{i 2\pi k \tau}\right)}$$

where $\tau \in \mathbb{H}$ so that $q = e^{i2\pi\tau}$ has $|q| < 1$ and the infinite product converges.

Topics this time

I Modular properties of η-function

II Combinatorics of partitions related to η-function

III Asymptotics of counting of partitions via modular properties of η-function

Recall from earlier talks:

- period lattice: $\Lambda = \mathbb{Z} + \tau\mathbb{Z}$ $\quad (\tau \in \mathbb{H})$
- modular group: ("group of orientation preserving automorphisms of Λ")

$$PSL(2,\mathbb{Z}) = \left\{ \tau \mapsto \frac{a\tau + b}{c\tau + d} \;\middle|\; a,b,c,d \in \mathbb{Z},\ ad - bc = 1 \right\}$$

$PSL(2,\mathbb{Z})$ generated by

$S : \tau \mapsto -1/\tau$

$T : \tau \mapsto \tau + 1$

- elliptic function:

$\varphi : \mathbb{C} \to \hat{\mathbb{C}}$ meromorphic

$\varphi(z + \omega) = \varphi(z) \quad \forall \omega \in \Lambda$

- modular function:

$f : \mathbb{H} \to \hat{\mathbb{C}}$ meromorphic

$f(M(\tau)) = f(\tau) \quad \forall M \in PSL(2,\mathbb{Z})$

f "meromorphic at $i\infty$"

(meaning: $q \mapsto f\left(\frac{1}{2\pi i}\log(q)\right)$ merom. at $q = 0$)

- entire modular form of weight $k \in \mathbb{Z}$:

$f : \mathbb{H} \to \mathbb{C}$ holomorphic

$$f(\underbrace{M(\tau)}_{\frac{a\tau+b}{c\tau+d}}) = \underbrace{M'(\tau)}_{(c\tau+d)^{k}}{}^{-k/2} \cdot f(\tau) \quad \forall M \in PSL(2,\mathbb{Z})$$

where we choose $c > 0$ (or $c = 0$, $d > 0$)

f "holomorphic at $i\infty$"

(meaning through $q = e^{i2\pi\tau}$ at $q = 0$)

Regarding modular properties of η, our main result will be:

<u>Theorem*</u> η would like to be a modular form

To make the statement more precise, we consider relaxing the conditions in the def. of entire modular form

<u>multiplier</u>: replace the covariance under $PSL(2,\mathbb{Z})$ by

$$f(M(\tau)) = M'(\tau)^{-k/2} \cdot \mu(M) \cdot f(\tau)$$

- There is an obvious consistency condition

$$f(M_1(M_2(\tau))) = f((M_1 \circ M_2)(\tau))$$

$$M_1'(M_2(\tau))^{-k/2} M_2'(\tau)^{-k/2} \mu(M_1)\mu(M_2) \cdot f(\tau) \qquad ((M_1 \circ M_2)'(\tau))^{-k/2} \mu(M_1 \circ M_2) \cdot f(\tau)$$

i.e. $\mu(M_1)\mu(M_2) = \mu(M_1 \circ M_2)$

- We can allow f to be vector valued and μ correspondingly matrix valued. The consistency condition says that $\mu: PSL(2,\mathbb{Z}) \to GL(\mathbb{C}^d)$ is a representation of the modular group.

<u>rational weights</u>: Instead of requiring $k \in \mathbb{Z}$ allow $k \in \mathbb{Q}$

However, in the consistency condition the step $((M_1 \circ M_2)')^{-k/2} = (M_1' \circ M_2)^{-k/2} \cdot (M_2')^{-k/2}$ involves branch choices. Now it only holds up to a certain root of unity factor.

$$\mu(M_1 \circ M_2) = \mu(M_1)\mu(M_2) \cdot c(M_1, M_2)$$

where $c: PSL(2,\mathbb{Z}) \times PSL(2,\mathbb{Z}) \to \mathbb{C}^\times$ is a 2-cocycle.

$$c(M_1, M_2M_3) \cdot c(M_2, M_3) = c(M_1M_2, M_3) \cdot c(M_1, M_2)$$

Thus μ is more like a projective representation.

<u>Theorem</u> η is a modular form of weight $k = 1/2$ and a non-trivial multiplier $\mu: PSL(2,\mathbb{Z}) \to \{24^{th}$ roots of $1\}$.

Concretely, under generators T and S we have

$\left(\zeta_{24} = e^{i\frac{\pi}{12}}\right)$

$$\eta(\tau+1) = \zeta_{24}\, \eta(\tau) \qquad \eta\left(\frac{-1}{\tau}\right) = \zeta_{24}^{-3} \sqrt{\tau}\, \eta(\tau).$$

Combinatorics of partitions ?

$$p(n) = \#\{(m_1, m_2, \dots, m_n) \in \mathbb{N}^n \mid \textstyle\sum_{j=1}^n j\cdot m_j = n\}$$
$$= \#\{(n_1, n_2, \dots, n_k) \mid k \in \mathbb{N},\ n_1 \geq n_2 \geq \dots \geq n_k \geq 1\}\quad \textstyle\sum_{i=1}^k n_i = n$$
$$= \text{number of partitions of } n \in \mathbb{N}$$

$$p^{\text{distinct}}_{\text{even/odd}} = \#\{(m_1, \dots, m_n) \in \{0,1\}^n \mid \textstyle\sum_{j=1}^n j\cdot m_j = n,\ \sum_{j=1}^n m_j \text{ even/odd}\}$$
$$= \#\text{ of partitions of } n \text{ to an even/odd number of distinct parts}$$

Let $\tau \in \mathbb{H}$ $\quad q = e^{i2\pi\tau}$ $\quad (|q|<1)$

<u>Theorem</u> (a) $$\frac{e^{i\frac{\pi}{12}\tau}}{\eta(\tau)} = \sum_{n=0}^\infty p(n)\, q^n =: \mathcal{P}(q)$$

(b) $$e^{-i\frac{\pi}{12}\tau}\eta(\tau) = \sum_{n=0}^\infty \left(p^{\text{distinct}}_{\text{even}}(n) - p^{\text{distinct}}_{\text{odd}}(n)\right) q^n =: \mathcal{P}^{\text{distinct}}_{\text{even}}(q) - \mathcal{P}^{\text{distinct}}_{\text{odd}}(q).$$

<u>Proof</u>: Easy, e.g.
$$\frac{q^{1/24}}{\eta(\tau)} = \prod_{j=1}^\infty (1-q^j)^{-1} = \prod_{j=1}^\infty \left(\sum_{m=0}^\infty q^{mj}\right) = \sum_{m_1, m_2, m_3, \dots} q^{\sum_{j=1}^\infty j\cdot m_j} = \mathcal{P}(q).$$
$$q^{-1/24}\cdot\eta(\tau) = \prod_{j=1}^\infty (1-q^j) = \sum_{m_1, m_2, \dots \in \{0,1\}} (-1)^{\sum m_j} q^{\sum j m_j} = \mathcal{P}^{\text{dist.}}_{\text{even}}(q) - \mathcal{P}^{\text{dist.}}_{\text{odd}}(q). \quad \square$$

Let's first see how easily modular properties give quite nontrivial results about the asymptotics of partitions.

<u>Corollary</u> (of modular properties of η) We have $p(n) \leq e^{\pi\sqrt{\frac{2}{3}n}}$.

<u>Proof</u>: Use the covariance of η under $S: \tau \mapsto \frac{-1}{\tau}$ with $\tau = i\frac{\delta}{2\pi}$, δ small.

$$\mathcal{P}(q) = \frac{q^{1/24}}{\eta(\tau)} = e^{i\frac{\pi}{12}\tau}\cdot\frac{1}{\eta(-1/\tau)}\zeta_{24}^{-3}\sqrt{\tau} = e^{i\frac{\pi}{12}(\tau+\frac{1}{\tau})}\mathcal{P}(q')\cdot\sqrt{\frac{\tau}{i}}$$
$q' = e^{2\pi i\frac{-1}{\tau}}$

$$\Longrightarrow \mathcal{P}(e^{-\delta}) = \sqrt{\frac{\delta}{2\pi}}\, e^{\frac{\pi^2}{6}\delta^{-1}}\cdot\underbrace{e^{-\frac{1}{24}\delta}\mathcal{P}(e^{-4\pi^2\delta^{-1}})}_{\longrightarrow 1 \text{ as } \delta\to 0} \leq e^{\frac{\pi^2}{6}\delta^{-1}}$$
for δ small.

Then $$p(n) \leq q^{-n}\sum_{k=0}^n p(k)\,q^k \leq q^{-n}\cdot\mathcal{P}(q) \leq \exp\left(n\delta + \frac{\pi^2}{6}\delta^{-1}\right).$$

Now optimize δ: choose $\delta = \frac{\pi}{\sqrt{6n}}$ to get $p(n) \leq e^{2\cdot\frac{\pi}{\sqrt{6}}\sqrt{n}} = e^{\pi\sqrt{\frac{2}{3}n}}$

$\square$

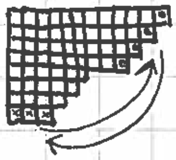
Let's not use the simplest way to prove modularity of η. Instead, the following way involves interesting properties of distinct partitions...

Theorem (Euler's pentagonal number theorem)

$$p^{\text{distinct}}_{\text{even}}(n) = p^{\text{distinct}}_{\text{odd}}(n) \qquad \text{if } n \neq \frac{k(3k\pm1)}{2}$$

and
$$p^{\text{distinct}}_{\text{even}}(n) = p^{\text{distinct}}_{\text{odd}}(n) + (-1)^k \qquad \text{if } n = \frac{k(3k\pm1)}{2}.$$

Proof [Franklin 1881]



$\square$

Corollary
$$\eta(\tau) = q^{\frac{1}{24}} \sum_{j \in \mathbb{Z}} (-1)^j \cdot q^{\frac{j\cdot(3j-1)}{2}}$$
$$= q^{\frac{1}{24}} \cdot \left(1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + q^{22} + q^{26} - q^{35} - \ldots\right)$$

We will next see how this implies modular properties of η.

Still some standard tools...

Fourier transform $\quad f : \mathbb{R} \to \mathbb{C}$ Schwarz class
$$\hat{f}(p) = \int_{\mathbb{R}} e^{-2\pi i p x} f(x)\, dx \qquad f(x) = \int_{\mathbb{R}} e^{2\pi i p x} \hat{f}(p)\, dp$$

Proposition (Poisson summation formula) $\displaystyle\sum_{j \in \mathbb{Z}} f(j) = \sum_{k \in \mathbb{Z}} \hat{f}(k)$.

Proof: Define periodic function $F(x) = \sum_{j \in \mathbb{Z}} f(x+j)$.

Do Fourier series $\quad F(x) = \displaystyle\sum_{k \in \mathbb{Z}} a_k e^{2\pi i k x}$

$$a_k = \int_0^1 e^{-i2\pi k x} F(x)\, dx = \sum_{j \in \mathbb{Z}} \int_0^1 e^{-i2\pi k x} f(x+j)\, dx = \int_{\mathbb{R}} f(x) e^{-i2\pi k x} dx = \hat{f}(k)$$

The equality follows by taking $x = 0$. $\square$

Lemma (Fourier transform of Gaussian)

If $f(x) = e^{-\alpha(x+b)^2}$ with $\mathrm{Re}(\alpha) > 0$ then $\hat{f}(p) = \sqrt{\frac{\pi}{\alpha}} \cdot e^{-\frac{\pi^2}{\alpha}p^2 + 2\pi i p b}$

Corollary
$$\sum_{j \in \mathbb{Z}} e^{-\alpha(j+b)^2} = \sum_{k \in \mathbb{Z}} \sqrt{\frac{\pi}{\alpha}} \cdot e^{-\frac{\pi^2}{\alpha}k^2 + 2\pi i k b}.$$

Proofs of modular transformation formulas for η:

T transformation trivial : $\eta(\tau+1) = e^{i\frac{\pi}{12}}\eta(\tau)$

$$\eta(\tau) = e^{i\frac{\pi}{12}\tau}\cdot\sum_{j\in\mathbb{Z}} e^{i2\pi\tau\cdot\frac{j\cdot(3j-1)}{2} + i\pi j}$$

(complete the square to use Lemma)

$$= e^{i\frac{\pi}{12}\tau}\sum_{j\in\mathbb{Z}}\exp\left(i\pi\tau\cdot 3\left(j-\frac{1-\tau^{-1}}{6}\right)^2 - i\pi\frac{\tau}{12}\left(1-\tau^{-1}\right)^2\right)$$

$$= e^{i\frac{\pi}{12}(+2-\tau^{-1})}\sqrt{\frac{-1}{3i\tau}}\sum_{k\in\mathbb{Z}}\underbrace{\exp\left(-i\frac{\pi}{3\tau}k^2 + 2\pi i k\frac{\tau^{-1}-1}{6}\right)}_{= e^{-i\frac{\pi}{3}k}\cdot e^{-i\frac{\pi}{3\tau}(k^2-k)}}$$

Consider separately contributions $k\equiv 0 \pmod 3$, $k\equiv 1 \pmod 3$, $k\equiv 2 \pmod 3$.

$k=3l+0$: $\displaystyle\sum_{l\in\mathbb{Z}} e^{-i\frac{\pi}{3\tau}(9l^2-3l)}(-1)^l$

$k=3l+1$: $\displaystyle\sum_{l\in\mathbb{Z}} e^{-i\frac{\pi}{3\tau}(9l^2+3l)}(-1)^l e^{-\frac{\pi i}{3}} = e^{-i\frac{\pi}{3}}\cdot(\text{previous})$

$k=3l-1$: $\displaystyle\sum_{l\in\mathbb{Z}} e^{-i\frac{\pi}{3\tau}(9l^2-9l+2)} e^{i\frac{\pi}{3}}(-1)^l = 0$ $\left(l \longleftrightarrow 1-l \text{ antisymmetry}\right)$

Therefore we have

$$\eta(\tau) = e^{-i\frac{\pi}{12}\tau^{-1}}\cdot 2\overbrace{\cos\left(\frac{\pi}{6}\right)}^{\sqrt{3}/2}\cdot\sqrt{\frac{+i}{3\cancel{i}\tau}}\cdot(\text{previous})$$

$$= \sqrt{\frac{i}{\tau}}\cdot e^{i\frac{\pi}{12}\frac{-1}{\tau}}\sum_{l\in\mathbb{Z}} e^{i2\pi\frac{3l^2-l}{2}\frac{-1}{\tau}} = \sqrt{\frac{i}{\tau}}\cdot\eta\left(\frac{-1}{\tau}\right)$$

$\therefore\quad \eta(T(\tau)) = \xi_{24}\cdot\eta(\tau)\,,\qquad \eta(S(\tau)) = \xi_{24}^{-3}\cdot\sqrt{\tau}\cdot\eta(\tau)$

branch: $\sqrt{\tau}>0 \;\; \forall\tau>0$

Since T, S generate $PSL(2,\mathbb{Z})$, it is now a matter of book-keeping of branch choices and 24^{th} roots of unity to obtain for any $M\in PSL(2,\mathbb{Z})$

$$\eta(M(\tau)) = \mu(M)\cdot M'(\tau)^{-1/4}\cdot\eta(\tau)$$

where $\mu(M)$ is some 24^{th} root of unity.

If you wish, there is a (useless looking) "explicit formula" for $\mu(M)$ involving Dedekind sums $s(h,k) = \sum_{r=1}^{k-1}\frac{r}{k}\left(\frac{hr}{k} - \left\lfloor\frac{hr}{k}\right\rfloor - \frac{1}{2}\right)$.

Theorem If $M = \begin{bmatrix} a & b \\ c & d\end{bmatrix}\in SL(2,\mathbb{Z})$ with $c>0$ then

$$\eta(M(\tau)) = e^{i\pi\left(\frac{a+d}{12} + s(-d,c) - \frac{1}{4}\right)}\sqrt{c\tau+d}\;\eta(\tau)$$

ON THE RADEMACHER SERIES

Number of partitions: $p(n) = \#\{(m_1, \dots, m_n) \in \mathbb{N}^n \mid \sum_{j=1}^{n} m_j \cdot j = n\}$

Generating function: $\displaystyle P(q) = \prod_{j=1}^{\infty} \frac{1}{1-q^j} = \frac{e^{i\frac{\pi}{12}\tau}}{\eta(\tau)}$ $\quad (q = e^{i2\pi\tau})$

where $\displaystyle \eta(\tau) = e^{i\frac{\pi}{12}\tau} \prod_{j=1}^{\infty}(1 - e^{i2\pi j\tau})$ is the Dedekind eta-function

Last time:

- $\eta(M(\tau)) = \mu(M) \cdot (M'(\tau))^{-1/4} \cdot \eta(\tau) \quad \forall M \in PSL(2,\mathbb{Z})$

  where the multiplier $\mu : PSL(2,\mathbb{Z}) \to \mathbb{C}^*$

- by considering $\tau \approx 0$ $(q \nearrow 1)$ and $M(\tau) = -1/\tau$

  we showed $p(n) \leq \exp\left(\pi\sqrt{\frac{2}{3}n}\right)$

We now want a convergent series for $p(n)$.

Idea: extract the contributions of all roots of unity $e^{2\pi i \frac{h}{k}}$, where $0 \leq \frac{h}{k} < 1$, $\gcd(h,k) = 1$.

$$p(n) = \frac{1}{2\pi i}\oint q^{-1-n} P(q)\, dq \qquad \text{(positively oriented closed contour encircling 0)}$$

$$= \int_C e^{-i2\pi n\tau} P(e^{i2\pi\tau})\, d\tau \qquad \text{(contour } C \text{ in half-plane } \mathbb{H} \text{ with end point = starting point + 1)}$$

Near the rational point $\frac{h}{k}$ we use M which maps $\frac{h}{k}$ to $i\infty$:

$$M(\tau) = \frac{\alpha\tau + \beta}{k\tau - h} \qquad (\text{just choose } \alpha, \beta \text{ so that } -\alpha h - \beta k = 1)$$

possible since $\gcd(h,k) = 1$

For $\tau = \frac{h}{k} + i\delta$:

$$M(\tau) = \frac{\alpha h + \beta k + i\alpha\delta k}{i\delta k^2} = \frac{i}{k^2}\delta^{-1} + \frac{\alpha}{k}$$

$$M'(\tau) = (i\delta k)^{-2}$$

$$P(e^{i2\pi\tau}) = e^{i\frac{\pi}{12}\tau}\frac{1}{\eta(\tau)} = e^{i\frac{\pi}{12}\tau}\frac{1}{\eta(M(\tau))}\mu(M)(M'(\tau))^{-1/4}$$

$$= \tilde{\xi}_{h,k} \cdot \sqrt{i\delta k}\; e^{i\cdot -\frac{\pi}{12}\delta}\; e^{-iM(\tau)} P(e^{i2\pi M(\tau)})$$

$$= \xi_{h,k}\sqrt{\delta k}\; e^{-\frac{\pi}{12}\delta + \frac{\pi}{12k^2}\delta^{-1}}\; \underbrace{P(\underbrace{e^{i2\pi M(\tau)}}_{\to 0 \text{ as } \delta \searrow 0})}_{\to 1 \text{ as } \delta \to 0}$$

Here and below we collect some root of unity factors in ξ.

$P(e^{i2\pi\frac{h}{k} - 2\pi\delta}) \sim e^{\frac{\pi}{12k^2}\delta^{-1}}$ "essential singularity" (milder for large k)

To make things look similar at different rational points $\frac{h}{k}$, we probe at distance $\delta \sim k^{-2}$.

For the part of C near $\frac{h}{k}$ we make the change of variables to z : $\tau = \frac{h}{k} + i\frac{z}{k^2}$ $\quad (z = O(1))$

and get an integral

$$\int_{C_{h/k}} e^{-i2\pi n\tau} P(e^{2\pi i\tau})\, d\tau$$

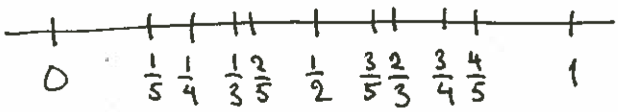
$$= k^{-5/2} \cdot \xi^{(n)}_{h,k} \int dz \, \exp\left\{\frac{z}{k^2} 2\pi\left(n - \frac{1}{24}\right) + \frac{\pi}{12}\frac{1}{zk^2}\right\} P(e^{i2\pi M(z)}) \sqrt{z}$$

So how do we build the contour C from $C_{h/k}$?

Farey fractions : We limit to rationals with denominator at most N, and in the end let $N \to \infty$.

$$F_N = \left\{ \frac{h}{k} \;\middle|\; h,k \in \mathbb{Z},\ 0 < k \le N,\ 0 \le h < k \right\}$$

we may assume $\gcd(h,k) = 1$.



Observations: (Always assume rationals given in lowest terms)

1°) For any $0 \le \frac{a}{b} < \frac{c}{d} \le 1$ the "mediant" $\frac{a+c}{b+d}$ lies between

$$\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$$

Pf: e.g. $\frac{c}{d} - \frac{a+c}{b+d} = \frac{cb - ad}{d(b+d)} > 0$ since $ad < bc$.

2°) Given $0 \le \frac{a}{b} < \frac{c}{d} \le 1$ such that "unimodularity" $bc - ad = 1$ holds, the points $\frac{a}{b}$ and $\frac{c}{d}$ are consecutive in F_N for $\max(b,d) \le N < b+d$.

Pf: "only if": clear.

"if": Suppose $\frac{a}{b} < \frac{h}{k} < \frac{c}{d}$. Then

$$k = k \cdot \underbrace{(bc - ad)}_{=1} = b \cdot \underbrace{(kc - hd)}_{>0} + d \cdot \underbrace{(bh - ak)}_{\ge 0} \quad (*)$$

3°) Given $0 \le \frac{a}{b} < \frac{c}{d} \le 1$ with unimod, we have for the mediant $\frac{h}{k} = \frac{a+c}{b+d}$ the unimod.: $kc - hd = 1$, $bh - ak = 1$

4°) $F_N \subset F_{N+1}$ and points in $F_{N+1} \setminus F_N$ are mediants of consecutive points of F_N.

5°) Consecutive Farey fractions satisfy unimodularity.

Ford circles

At rational pt $\frac{h}{k}$ the Ford circle is the circle with center $\frac{h}{k} + i\frac{1}{2k^2}$ and radius $\frac{1}{2k^2}$.

Observations

1°) Two Ford circles at $\frac{a}{b}$ and $\frac{c}{d}$ are either tangent to each other or don't intersect. They are tangent iff $|bc-ad|=1$ (i.e. if $\frac{a}{b}$ and $\frac{c}{d}$ are consec. Farey frac

Pf: (dist of centers)^2 − (radius + radius)^2

$$= \left\{\left(\frac{c}{d}-\frac{a}{b}\right)^2 + \left(\frac{1}{2b^2}-\frac{1}{2d^2}\right)^2\right\} - \left(\frac{1}{2b^2}+\frac{1}{2d^2}\right)^2$$

$$= \frac{(cb-ad)^2}{(bd)^2} - 4\cdot\frac{1}{2b^2\cdot 2d^2} = \frac{(bc-ad)^2-1}{b^2d^2}$$

2°) If $\frac{h_1}{k_1} < \frac{h}{k} < \frac{h_2}{k_2}$ are consecutive Farey fractions, then the points of contact have imaginary parts $\frac{1}{k^2+k_1^2}, \frac{1}{k^2+k_2^2}$.

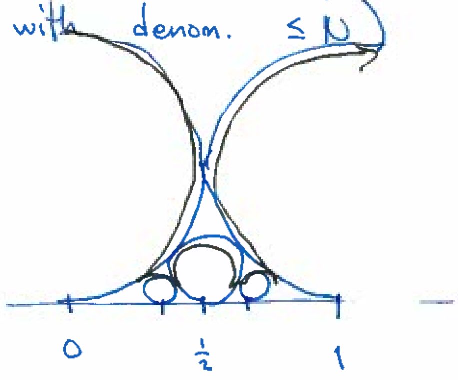
Pf: easy trigonometry.

3°) For a given $\frac{h}{k}$, as $N \to \infty$, the points of contact of the Ford circle with the Ford circles of consecutive Farey factions in F_N have imaginary parts $\Theta\left(\frac{1}{N^2}\right)$ and distance to $\frac{h}{k}$ of $\Theta\left(\frac{1}{Nk}\right)$.

Pf: $y \sim \left(\left(x-\frac{h}{k}\right)\cdot k\right)^2$.

4°) After the change of variables $\tau = \frac{h}{k} + i\frac{z}{k^2}$ to z, the entire chord connecting the points of contact is within distance $\Theta\left(\frac{k}{N}\right)$ from $z=0$.

Fix n. Construct Rademacher path of integration from the upper arcs of Ford circles (at Farey fractions with denom. $\leq N$)



$$C = \bigcup_{k=1}^{N} \bigcup_{\substack{0 \leq h < 1 \\ \gcd(h,k)=1}} C_{h,k}^{(N)}$$

arc of Ford circle

The change of variables to z takes this to an arc of circle of radius $\frac{1}{2}$ centered at $\frac{1}{2}$.

$$p(n) = \sum_{k=1}^{N} k^{-5/2} \sum_{h} \xi_{h/k}^{(n)} \int dz \sqrt{z} \cdot \exp\left\{ \frac{z}{k^2} 2\pi (n - \tfrac{1}{24}) + \frac{1}{zk^2} \frac{\pi}{12} \right\} \underbrace{P(e^{\cdots})}_{\approx 1}$$

Approximations

- Replace $P(e^{i2\pi M(\tau)})$ by 1.
  The error in the integrand is $O(|z|^{1/2})$
  The cumulative error is (do integration along the chord which is within $O(k/N)$ of 0)

$$\underbrace{\sum_{k=1}^{N}}_{\#\text{ terms} = O(N)} k^{-5/2} \underbrace{\sum_{h}}_{\#\text{ terms} = O(k)} \underbrace{O\left(\frac{k}{N}\right)}_{\text{length of chord}} \cdot \underbrace{O\left(\sqrt{\frac{k}{N}}\right)}_{\text{error of integrand}} = O(N^{-3/2}) \sum_{k=1}^{N} \underbrace{\sum_{h} k^{-1}}_{O(1)} = O(N^{-1/2})$$

- Replace the integration contour, an arc of the circle, (radius $\frac{1}{2}$, centre $\frac{1}{2}$), by the entire circle.
  The integrand is $O(|z|^{1/2}) = O\left(\sqrt{\frac{k}{N}}\right)$ uses $\mathrm{Re}(\frac{1}{z}) = 1$
  and the segment lenghts are $O(\frac{k}{N})$, so the total error is again $O(N^{-1/2})$.

$z = \frac{1}{2}(1 + e^{i\theta})$

$\frac{1}{z} = \frac{2(1+e^{-i\theta})}{|(1+e^{i\theta})|^2}$

$= 2 \frac{1 + \cos\theta + i \cdots}{2 + 2\cos\theta}$

$= 1 + \# i$

$$\therefore \quad \boxed{p(n) = \sum_{k=1}^{N} k^{-5/2} \sum_{h} \xi_{h,k}^{(n)} \int_{\partial B(\frac{1}{2}, \frac{1}{2})} dz \sqrt{z} \exp\left(\frac{z}{k^2} 2\pi (n - \tfrac{1}{24}) + \frac{1}{zk^2} \frac{\pi}{12} \right) + O\left(\frac{1}{\sqrt{N}}\right)}$$

Rademacher series

Can be simplified: the integral gives a Bessel function

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} \left(\sum_{h} \xi_{h,k}^{(n)} \right) k^{1/2} \frac{d}{dn} \left(\frac{\sinh\left(\frac{\pi}{k} \sqrt{\frac{2}{3}(n - 1/24)} \right)}{\sqrt{n - 1/24}} \right)$$