Homological algebra

Chain complexes

Chain complex (C, d) consists of

- An abelian group C_n for every $n \in \mathbb{Z}$
- A homomorphism $d_n \colon C_n \to C_{n-1}$ for every $n \in \mathbb{Z}$.

The only thing that is required is that $d_{n-1} \circ d_n = 0$ for every $n \in \mathbb{Z}$.

Complex is non-negative if $C_n = 0$ for $n < 0$ and free if C_n is free abelian group for all $n \in \mathbb{Z}$. All important chain complexes we use in this course, such as $C(K, L)$ and $C(X, A)$ are free and non-negative.

- Group of *n-cycles* in chain complex C is
$$Z_n(C) = \operatorname{Ker} d_n \subset C_n$$
- Group of *n-boundaries* in chain complex C is
$$B_n(C) = \operatorname{Im} d_{n+1} \subset C_n$$
- Because of $d_n \circ d_{n+1} = 0$, $B_n(C) \subset Z_n(C)$, so we can form **homology groups**
$$H_n(C) = Z_n(C)/B_n(C).$$

A homology class of a cycle $z \in Z_n(C)$ is its equivalence class $\bar{z}$ in the quotient group $H_n(C)$. Two cycles z, z' define the same homology class if $z - z'$ is a boundary i.e. $z - z' = d_{n+1}(u)$ for some $u \in C_{n+1}$.

Chain complex C is **acyclic** if $H_n(C) = 0$ for all $n \in \mathbb{Z}$.

Chain mappings

Suppose (C, d) and (C', d') are chain complexes. A collection f of homomorphisms $f_n \colon C_n \to C'_n$ is a *chain mapping* if it commutes with boundary operators i.e. if $d'_n \circ f_n = f_{n-1} \circ d_n$ for all $n \in \mathbb{Z}$.

Chain mapping maps cycles to cycles and boundaries to boundaries, so **induces homomorphisms** $f_* \colon H_n(C) \to H_n(C')$ for all $n \in \mathbb{Z}$, defined by

$f_*(\bar{z}) = \bar{f(z)}$.

Chain mappings can be composed - if $f\colon C \to C'$ and $g\colon C' \to C''$ are chain mappings, there exist chain mapping $g \circ f$ defined as an ordinary composition "componentwise". There exists identity chain mapping $\mathrm{id}\colon C \to C$. A chain mapping $f\colon C \to D$ is an *isomorphism* if an only if $f_n\colon C_n \to D_n$ is an isomorphism for all $n \in \mathbb{Z}$.

"Star"-operator $f \mapsto f_*$ respects identity mapping and composition of chain mappings, i.e. $\mathrm{id}_* = \mathrm{id}$, $(g \circ f)_* = g_* \circ f_*$.

A **subcomplex** C' of a chain complex C is defined in a natural way. Then there exists an inclusion chain mapping $i\colon C' \to C$, **quotient complex** C/C' and chain projection mapping $p\colon C \to C/C'$. Kernel and image of a chain mapping are chain subcomplexes of corresponding chain complexes. There exists analogue of Factorization Theorem as well as the Isomorphism Theorem for chain complexes and chain mappings (Proposition 10.8. and Corollary 10.9).

Direct sum $C = \oplus_{\alpha\in\mathcal{A}} C_\alpha$ of chain complexes C_α is defined in a natural way "componentwise". The homology groups of C are isomorphic to the direct sum of corresponding homology groups of C_α,

$$H_n(\oplus_{\alpha\in\mathcal{A}} C_\alpha) \cong \oplus_{\alpha\in\mathcal{A}} H_n(C_\alpha).$$

Exact sequences of abelian groups

Suppose we have a sequence

$$\dots \longrightarrow A_{n+1} \xrightarrow{f_{n+1}} A_n \xrightarrow{f_n} A_{n-1} \longrightarrow \dots$$

of abelian groups and homomorphisms. It can be unlimited in both direction, i.e. indexed on the set of integers $\mathbb{Z}$, or stop somewhere on the left or/and on the right. We say that this sequence is **exact** at A_n if

$$\mathrm{Ker}\, f_n = \mathrm{Im}\, f_{n+1}$$

provided, that the mappings f_n and f_{n+1} are defined. If the sequence is exact at every group A_n that appears in it, we say that the sequence is an **exact sequence** of abelian groups and homomorphisms.
An exact sequence of abelian groups and homomorphisms is the same thing as acyclic chain complex.

The exact sequence is **short exact** if it is of the form

$$0 \longrightarrow A \xrightarrow{f} C \xrightarrow{g} B \longrightarrow 0$$

This is equivalent to

- f is injection,
- g is surjection,
- $\operatorname{Im} f = \operatorname{Ker} f$.

This implies that every short exact sequence is essentially of the form

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{p} B/A \longrightarrow 0$$

where i is inclusion of the subgroup and p is canonical projection to a quotient group (Lemma 11.4).

Short exact sequence

$$0 \longrightarrow A \xrightarrow{f} C \xrightarrow{g} B \longrightarrow 0$$

splits if it is essentially isomorphic to the sequence of the form

$$0 \longrightarrow A \xrightarrow{i} A \oplus B \xrightarrow{p} B \longrightarrow 0,$$

where $i(a) = (a, 0)$ and $p(a, b) = b$. Precise definition in 11.15. Lemma 11.16 gives important equivalent characterizations which are usually easier to apply in practise.

Five Lemma 11.4. is an important tool with a lot of applications.

Short exact sequences of chain complexes and induced long exact homology sequence

The sequence

$$0 \longrightarrow C' \xrightarrow{f} C \xrightarrow{g} \overline{C} \longrightarrow 0 \qquad (0.1)$$

of **chain complexes and chain mappings** is **short exact** if it is short exact sequence of abelian groups and homomorphisms "in every dimension", more precisely if the sequence

$$0 \longrightarrow C'_n \xrightarrow{f_n} C_n \xrightarrow{g_n} \overline{C}_n \longrightarrow 0$$

is short exact sequence of abelian groups and homomorphisms for all $n \in \mathbb{Z}$.

One of the main results of homological algebra is the existence of **long exact homology sequence**

$$\ldots \longrightarrow H_{n+1}(\overline{C}) \xrightarrow{\Delta_{n+1}} H_n(C') \xrightarrow{f_*} H_n(C) \xrightarrow{g_*} H_n(\overline{C}) \xrightarrow{\Delta_n} H_{n-1}(C') \longrightarrow \ldots$$

induced by the short exact sequence (0.1). The boundary operator $\Delta\colon H_n(\overline{C}) \to H_{n-1}(C)$ is defined by the following algorithm. Let $\bar{z} \in H_n(\overline{C})$ be a homology class of a cycle $z \in Z_n(\overline{C})$. Then $\Delta(\bar{z}) = \bar{x} \in H_{n-1}(C)$ for the cycle x such that $f_{n-1}(x) = d_n y$, where $y \in C_n$ is such that $g_n(y) = z$.

The long exact homology sequence is "natural". Precisely put it means that if we have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C' & \xrightarrow{f} & C & \xrightarrow{g} & \overline{C} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\
0 & \longrightarrow & D' & \xrightarrow{f'} & D & \xrightarrow{g'} & \overline{D} & \longrightarrow & 0
\end{array}$$

of chain complexes and chain mappings with short exact rows. Then the induced diagram between induced long exact homology sequences

$$\begin{array}{ccccccccccccc}
\ldots & \longrightarrow & H_{n+1}(\overline{C}) & \xrightarrow{\Delta_{n+1}} & H_n(C') & \xrightarrow{f_*} & H_n(C) & \xrightarrow{g_*} & H_n(\overline{C}) & \xrightarrow{\Delta_n} & H_{n-1}(C') & \longrightarrow & \ldots \\
 & & \downarrow{\scriptstyle \gamma_*} & & \downarrow{\scriptstyle \alpha_*} & & \downarrow{\scriptstyle \beta_*} & & \downarrow{\scriptstyle \gamma_*} & & \downarrow{\scriptstyle \alpha_*} & & \\
\ldots & \longrightarrow & H_{n+1}(\overline{D}) & \xrightarrow{\Delta_{n+1}} & H_n(D') & \xrightarrow{f'_*} & H_n(D) & \xrightarrow{g'_*} & H_n(\overline{D}) & \xrightarrow{\Delta_n} & H_{n-1}(D') & \longrightarrow & \ldots
\end{array}$$

is also commutative.

An important algebraic application of long exact homology sequence can be found in Proposition 11.11. It has been applied a lot when switching from special absolute case to reduced or relative cases. For a typical application of Proposition 11.11 see the proof of existence of long exact reduced homology sequence (pages 191-192) or the proof that there exists relative/reduced version of Mayer-Vietoris for proper triads (Exercise 12.1).

Augmentation and reduced homology

Augmentation in a non-negative chain complex C is a surjective homomorphism $\varepsilon_0\colon C_0 \to \mathbb{Z}$ such that $\varepsilon \circ d_1 = 0$. An equivalent definition augmentation is a chain mapping $\varepsilon\colon C \to \mathbb{Z}_C$, where $\mathbb{Z}_C$ is a chain complex with $(\mathbb{Z}_C)_0 = \mathbb{Z}$ and other groups trivial. **Reduced chain complex** $\tilde{C}$ is defined as a kernel of augmentation. **Reduced homology groups** $\tilde{H}_n(C)$ are homology groups of this complex $\tilde{C}$.

Reduced groups have the following relation with ordinary homology groups,

$$\widetilde{H}_n(C) = H_n(C) \text{ if } n \neq 0,$$

$$H_0(C) \cong \widetilde{H}_0(C) \oplus \mathbb{Z} \text{ and}$$

$$\widetilde{H}_0(C) = \operatorname{Ker} \varepsilon_*.$$

In particular $\tilde{H}_n(C)$ is a subgroup of $H_n(C)$.

A complex C equipped with its augmentation is called augmented chain complex. A chain mapping $f\colon C \to C'$ between two augmented chain complexes C, C' *preserves augmentation* if $\varepsilon' \circ f = \varepsilon$. Such a mapping induce homomorphisms $f_*\colon \tilde{H}_n(C) \to \tilde{H}_n(C')$ for all $n \in \mathbb{Z}$, which are just restrictions of $f_*\colon H_n(C) \to H_n(C')$.

Suppose

$$0 \longrightarrow C' \xrightarrow{f} C \xrightarrow{g} \overline{C} \longrightarrow 0$$

is a short exact sequence of chain complexes such that C, C' are augmented and f preserves augmentation. Then there exists **long exact reduced homology sequence**

(0.2)

$$\ldots \longrightarrow H_{n+1}(\overline{C}) \xrightarrow{\Delta} H_n(C') \xrightarrow{f_*} H_n(C) \xrightarrow{g_*} H_n(\overline{C}) \xrightarrow{\Delta} H_{n-1}(C') \longrightarrow \ldots$$

$$\ldots \longrightarrow H_1(\overline{C}) \xrightarrow{\Delta} H_0(\widetilde{C'}) \xrightarrow{f_*} H_0(\widetilde{C}) \xrightarrow{g_*} H_0(\overline{C}) \longrightarrow 0 \qquad .$$

Reduced homology groups are usually easier to work with, compared to ordinary homplogy groups

Chain homotopy

Suppose $\alpha, \beta\colon C \to C'$ are chain mappings between chain complexes. The collection $H = (H_n)_{n\in\mathbb{N}}$ of homomorphisms $H_n\colon C_n \to C'_{n+1}$ is called **a chain homotopy** between α and β if

$$d'_{n+1}H_n + H_{n-1}d_n = \alpha_n - \beta_n,$$

for all $n \in \mathbb{Z}$.

If for chain mappings $\alpha, \beta\colon C \to C'$ there exists a chain homotopy H between them, we say that α and β are **chain homotopic**.

The main reason chain homotopy is interesting is the fact that **chain homotopic mappings induce the same mappings in homology** i.e. $\alpha_* = \beta_*\colon H_n(C) \to H_n(C')$. The same is true for reduced groups, if complexes are augmented and both α, β preserve augmentation.

Mayer-Vietoris sequence

Suppose C, C' are subcomplexes of a chain complex D. Then there exist short exact sequence

$$0 \longrightarrow C \cap C' \xrightarrow{h} C \oplus C' \xrightarrow{q} C + C' \longrightarrow 0,$$

$$h_n(x) = (x, -x),$$

$$q_n(x, y) = x + y.$$

The induced long exact homology sequence

$$\ldots \longrightarrow H_{n+1}(C + C') \xrightarrow{\Gamma} H_n(C \cap C') \xrightarrow{h_*} H_n(C \oplus C') \xrightarrow{q_*} H_n(C + C') \longrightarrow \ldots$$

is called **Mayer-Vietoris sequence** of the pair (C, C'). It can be used to calculate the homology of $C + C'$, whenever homologies of C, C' and $C \cap C'$ are known. In general $C + C'$ is not necessarily interesting but if inclusion $C + C' \to D$ induces isomorphisms in homology, Mayer-Vietoris gives a way to calculate $H_n(D)$.