Foreword

Although "mathematics" in general is one of the oldest form of scientific reasoning human race has been exercising, from the formal point of view modern mathematics based on the abstract set-theoretical approach is quite a modern science. It has started to take shape in 19th century, when likes of Cauchy, Riemann, Cantor, to name only a few, started to develop foundations for mathematics.

Partly this development was motivated by the fact that at that point researchers of mathematics started to stumble upon important mathematical questions, which seemed to be very difficult, or even impossible, to answer using traditional "intuitive" approach, that seemed to suffice before.

As an example, consider Euclidean vector spaces $\mathbb{R}^n$. One cannot underestimate the importance of the set of real numbers $\mathbb{R}$, and, as a consequence, the importance of finite-dimensional Euclidean spaces, in mathematics and its real-life applications. We stumble upon real numbers and vectors (finite strings of real numbers) continuously - along with more basic concept of natural number, real numbers seem to be one of the fundamental concepts of mathematical science. Hence it is extremely important to understand Euclidean spaces and their relationships as well as we can. It seems somehow "intuitively clear", that different spaces $\mathbb{R}^n$ and $\mathbb{R}^m$, $m \neq n$ should not be homeomorphic as topological spaces, but it took many years of research and the development of advanced mathematical tools and techniques to actually prove this result, known as " The invariance of domain " principle, precisely.

Perhaps one of the reason this claim seems obvious is our intuition regarding the notion of " dimension ". For example, it seems clear that the plane $\mathbb{R}^2$ has "more space" as the real line $\mathbb{R}$ and the 3-dimensional " space " $\mathbb{R}^3$ has even more "filling" than the plane, so it should be impossible to even fill bigger dimensional space with smaller dimensional, let alone for them to be geometrically "similar" (which is what homeomorphic means). However, this stronger claim, which is supported by our intuition, is not true at all. Around 20 years before the invariance of domain was actually proven by L.E.J. Brouwer, italian mathematician Giuseppe Peano managed to constuct a surjective continuous mapping $f\colon I \to I^2$ (also known as " the space-filling curve "), thus showing that you can actually " fill " bigger dimensional object with smaller dimensional. Here $I = [0, 1]$ is a closed interval. Before that the inventor of the set theory (which today is recognised as the theory that provides foundations for the formal mathematics) Georg Cantor has shown that there is a bijection mapping $F\colon \mathbb{R} \to \mathbb{R}^2$, which means that the real line has "the same amount of points " as the plain. Such a bijection is, luckily, could never be continuous, but nevertheless all these result are considered "counter-intuitive" from the naive point of view, so before the invariance of domain was actually proved, one could even suspected if it was actually false.

Another interesting aspect of the invariance of domain is that it is a very elementary, basic question. It is very easy to formulate and a freshman studying mathematics would easily understand the problem completely. So it seems like the proof of it should also be not very difficult, but surprisingly it turned out that it is. Many false or incomplete attempts to solve this problem and similar problems were made by quite famous mathematicians before the flawless proof finally emerged. One of the reasons for that is the fact that this claim is actually very general - it asserts that among the set of all the continuous mappings $f\colon \mathbb{R}^n \to \mathbb{R}^m$ one cannot find a single homeomorphism. It is quite easy to prove the claim that one cannot find a homeomorphism among differentiable mapping, or otherwise mappings having "regular geometrical behaviour". Unfortunately this is nearly not enough, because there are plenty of continuous mappings $f\colon \mathbb{R}^n \to \mathbb{R}^m$ that are not "regular" and, as Piano's space-filling curve has shown, even a plenty of surjective continuous mappings $f\colon \mathbb{R}^n \to \mathbb{R}^m$. It turns out extremely difficult to classify all these mappings and, as mentioned before, most of them turn out not to have regular behaviour - meaning to be for instance differentiable or otherwise possible to "visualise" somehow. For instance it is possible to show that "most" continuous mappings $f\colon \mathbb{R}^n \to \mathbb{R}^m$ are not differentiable **anywhere**. So, how can one show precisely that there could not be some weird pathologically looking homeomorphism, say, $\mathbb{R}^4 \to \mathbb{R}^7$? The fact that geometrical intuition cannot be used directly in spaces with dimension ≥ 4 makes the claim feel even less obvious. It is easy for humans to see distinction between $\mathbb{R}^2$ and $\mathbb{R}^3$, but how do we really know that $\mathbb{R}^{25}$ should be different from $\mathbb{R}^{26}$? Is there really such a big difference and how do we find it? Where do we even search for it?

Pretty soon the mathematicians that had tried to solve such difficult topological problems, realized that the right strategy lies in the construction of **invariants** i.e. "objects" that are associated to spaces and mappings of spaces and somehow reflect their properties. If these invariants are somehow " simpler " than the studied space i.e. reflect only some of its properties, they are easier to handle, so the difficult problem might turn into a simpler problem, concerning these invariants, which is possible to solve. Today most of such invariants are algebraic in nature, that is why the field in now known as **algebraic** topology. In the beginning, for instance in Brouwer's days, many invariants were merely discrete objects, for example integer numbers, that is why then the approach was known as the **combinatorial** topology.

So, in a nutshell, the idea of algebraic topology is as follows. In topology we study geometrical objects which are formalized to be so-called topological spaces and we also study mappings between them, which preserve the topological structure. Such mappings are what is known to be continuous mappings. To every topological space X, which we wish to study, we attach a corresponding algebraic object, let us denote it by $F(X)$. It can be a group, a ring, a field, a vector space, or anything else from the world of algebra. The functional notation $F(X)$ serve to remind us that the invariant $F(X)$ depends on the space.

But these are not enough. As the mappings between spaces are equally important[1], we also attach an algebraic mapping $F(f)\colon F(X) \to F(Y)$ to any continuous mapping $f\colon X \to Y$ (or, more generally, to any continuous mapping which we consider important enough to study). That is, for example, if all $F(X)$ are groups, then $F(f)$ would be a homomorphism of groups. If $F(X)$'s are vector spaces, $F(f)$ would be linear mappings, and so on.
We also require the following natural properties (known as the *functorial properties*)[2].

(1) If $f\colon X \to Y$ and $g\colon Y \to Z$ are continuous mapping between spaces, we have that
$$F(g \circ f) = F(g) \circ F(f).$$
That is, our invariants commute with the composition of mappings.

(2) Consider the identity mapping $id\colon X \to X$. Then $F(\mathrm{id})\colon F(X) \to F(X)$ is also an identity mapping.

How would this properties help us? Suppose we have two spaces X and Y and we wish to study if they are homeomorphic or not. In the special case $X = \mathbb{R}^n$, $Y = \mathbb{R}^m$ this is precisely invariance of domain problem. Suppose they are, and let $f\colon X \to Y$ be a homeomorphism. By definition, this means that there exists continuous mapping $g\colon Y \to X$ (inverse of f) such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$. If we apply invariant F to these equations, we obtain (using functorial properties (1) ad (2) above) similar equations
$$F(g) \circ F(f) = F(g \circ f) = F(\mathrm{id}_X) = \mathrm{id}_{F(X)},$$

[1] From the point of view of categorically wired mathematician even more important.

[2] To be precise these are what are known to be *covariant* functorial properties. There are also many contravariant invariants in algebraic topology, which means that instead of $F(f)\colon F(X) \to F(Y)$ we have a mapping $F(f)\colon F(Y) \to F(X)$ in another direction, but let us not concern ourselves with such technical details at this point.

$$F(f) \circ F(g) = F(f \circ g) = F(\mathrm{id}_Y) = \mathrm{id}_{F(Y)} .$$

In particular this would imply that $F(f)\colon F(X) \to F(Y)$ is an isomorphism between algebraic objects $F(X)$ and $F(Y)$. Now, turn this upside down. Suppose we do not know if X and Y are homeomorphic, but we are able to construct an algebraic invariant F such that $F(X)$ and $F(Y)$ **are not** isomorphic. Then, by above reasoning, we see immediately that X and Y are not homeomorphic, so our problem is solved.
Hence algebraic topology provides us with tools which enables us to translate topological problems into algebraic problems.

Of course this sounds simple only in theory. In practise the development of such tools is a hard work. First of all, this translation process should be able to translate our difficult problem (which we cannot solve directly) into a much simpler algebraic problem - which we can solve. Otherwise, our tool is useless. So it means that invariant F must simplify spaces and maps it is applied to, hence it must lose a lot of information, but still preserve enough of it, so that the problems could be solved. For instance, if using invariant in a manner explained in a previous paragraph, we obtain isomorphic algebraic objects $F(X)$ and $F(Y)$, this does not tell us anything about our original problem, since in theory non-homeomorphic objects could also produce isomorphic invariants. Actually, since a useful invariant must lose some information, such situations are actually even expected. So, this also implies that one invariant could not suffice. But if we are able to construct many different invariants, then there is a hope that whenever we have a topological problem, one of the invariants available would "lock in" on our problem, just like a suitable size wrench key would lock on a pipe we wish to turn around. And the more different invariants we have, the more problems we can solve using them.

Hence a much accurate explanation of what algebraic topology does and what is for is the following. Algebraic topology is concerned with the construction of different useful algebraic invariants for the purpose of study of topological problems. The main objective of this introductory course is to provide an important and widely used example of such a construction. Precisely put, we shall go through the classical singular homology theory, which give us a useful family of algebraic invariants, called singular homology groups. We will also show how singular homology theory is applied in order to prove the classical topological problems such as the invariance of domain. Other similar problems we will investigate include the following:

(1) Invariance of domain, general version - if U and V are homeomorphic

subsets of $\mathbb{R}^n$ and U is open, also V is open.

(2) Brouwer-Jordan separation theorem - if $S \subset \mathbb{R}^n$ is homeomorphic to the sphere S^{n-1}, then $\mathbb{R}^n \setminus S$ has exactly two path components and S is a boundary of both.

(3) Brouwer fixed point theorem - any continuous mapping $f\colon \overline{B}^n \to \overline{B}^n$ has a fixed point i.e. $f(x) = x$ for some $x \in \overline{B}^n$.

(4) The sphere S^n is not contractible to a point.

(5) The sphere S^n is not a retract of the disk $\overline{B}^{n+1}$.

(6) Hairy Ball Theorem - if n is even, the sphere S^n has no non-zero tangent vector field.

The singular homology theory itself is a relatively modern construction, it was invented in 1940's. All the claims listed above very already proved at that point, using earlier versions of the similar ideas, such as Betti's numbers and various simplicial methods. The latter has not only historical value - simplicial methods are still very usable and important in the modern mathematics, both in theory as well as in the concrete calculations and applications. The singular homology theory itself is actually based on the usage of simplices. Simplices have also given rise to a number of abstract generalizations, such as simplicial objects and related abstract combinatorical notions. That is why we start the course with the brief journey to the more geometrical and less abstract world of simplices and simplicial methods. This part of the course does not contain any algebra and is intended to give a reader the chance to see some concrete and more down-to-earth mathematics related to our main subjects, before diving into abstract algebra of homology theory. On the other hand geometric notions of this introduction will make it easier to understand and motivate the more abstract homological algebra which constitutes the main content of the course.

After some necessary technical algebraic tools, such as the theory of chain complexes, are developed, we define and study the properties of the singular homology theory. After this machinery is complete, we apply it to topological problems such as the ones listed above. The end of that part is dedicated to the study of a very important notion, which is the degree of a mapping $f\colon S^n \to S^n$. Historically this notion and simplicial approximation were precisely the tools Brouwer used to prove his fixed point theorem and invariance of domain theorem in the beginning of the 20th century. We define the concept of the degree using the singular homology theory. Of course Brouwer did not know anything about homology groups, so his definition was more complicated and geometrical in nature. However, it was one of the first examples of the combinatorial invariants defined for topological objects.

The last part of the course will be a journey into the world of CW-complexes and cellular homology. We will also go through the classification of compact 2-manifolds. The special case of this classification theorem gives a two-dimensional version of the famous "Poincare conjecture". Roughly speaking, "Poincare conjecture" says that any compact 3-mainfold which is homotopy equivalent to the sphere S^3, is actually homeomorphic to. This problem has remained unsolved for approximately 100 years, until it was recently proved by Russian mathematician G. Perelman. Of course, we won't be going through the proof of the "Poincare conjecture", but through much more modest and simple 2-dimensional version of it, which is a classical result.

It goes without saying that in this course we only have opportunity to scratch the very surface of the subject known as " algebraic topology ", that is why it is called "introduction to the algebraic topology". Perhaps the more precise name would be " introduction to homological methods", since it is mainly homology we introduce ourselves to. Another big branches of algebraic topology include cohomology theory, homotopy theory, K-theory, theory of obstructions and many others. Even the homology theory itself has much more into it then what we will see in this course. The interested reader is advised to continue the studies of algebraic topology by consulting the books that are listed below. These books are also recommended as the additional reading throughout the course.

References

[1] Eilenberg, S., Steenrod, N.: *Foundations of Algebraic Topology* - Princeton University Press, 1952.

[2] Hatcher, A.: *Algebraic Topology* - http://www.math.cornell.edu/ hatcher/AT/AT.pdf, 2002.

[3] Maunder, C.R.F: *Algebraic topology* - Van Nostrand Reinhold Company, 1970.

[4] Spanier, E.H: *Algebraic Topology* - McGraw-Hill, 1966.