Department of Mathematics and Statistics
Introduction to Algebraic topology, fall 2013
Exam 2 FR 13.12.2013 Solutions

1. Suppose

$$0 \longrightarrow C' \xrightarrow{f} C \xrightarrow{g} \overline{C} \longrightarrow 0 \tag{0.1}$$

is a short exact sequence of chain complexes and chain mappings.
a) Explain how the boundary operator $\Delta\colon H_n(\overline{C}) \to H_{n-1}(C')$ induced by (0.1) is constructed and show that it is well-defined.
b) Prove that the sequence

$$H_n(C') \xrightarrow{f_*} H_n(C) \xrightarrow{g_*} H_n(\overline{C})$$

is exact at $H_n(C)$.

Solution: We denote boundary operators of C' by d', boundary operators of C by d and boundary operators of $\overline{C}$ by $\overline{d}$.

a) The algorithm for the definition of Δ is the following. Suppose $\overline{z} \in H_n(\overline{C})$ is a homology class of a cycle $z \in Z_n(\overline{C})$. First, using the fact that $g_n\colon C_n \to \overline{C}_n$ is a surjection (this follows by exactness of 0.1), we pick $y \in C_n$ such that $g_n(y) = z$. Since g is a chain mapping we then have that

$$g_{n-1}d_n(y) = \overline{d}_n(g_n(y)) = \overline{d}_n(z) = 0,$$

since z is a cycle in $\overline{C}$. Hence $d_n(y) \in \operatorname{Ker} g_n = \operatorname{Im} f_{n-1}$ (the last equation is true by exactness of 0.1), so there exists $x \in C'_{n-1}$ such that $f_{n-1}(x) = d_n(y)$. Moreover, since f_{n-1} is injective, such an element x is **unique**. Next we show that such an element x is always a cycle element in C'_n. Using the fact that f is a chain mapping, we obtain

$$f_{n-2}d'_{n-1}(x) = d_{n-1}f_{n-1}(x) = d_{n-1}d_n(y) = 0,$$

since C is a chain complex. Since f_{n-2} is injective (by exactness of 0.1), $d'_{n-1}(x) = 0$, so $x \in Z_{n-1}(C')$. In particular the homology class $\overline{x} \in H_{n-1}(C')$ exists.

Next step is to show that the rule that assigns to $z \in Z_n(\overline{C})$ an element $\overline{x} \in H_{n-1}(C')$ obtained by the algorithm explained above is a well-defined function $\delta\colon Z_n(\overline{C}) \to \overline{x} \in H_{n-1}(C')$. So far we have shown that for every $z \in Z_n(\overline{C})$ an element x constructed by the procedure above is always a cycle, so the homology class $\overline{x} \in H_{n-1}(C')$ exists. However the procedure involved the choice of $y \in C_n$, which is, in general, not unique (unless g is also injective, in which case C' must be zero complex). Hence, we need to show that $\overline{x}$ does not depend on the choice of y above. Suppose $y' \in C_n$ is another choice i.e. $g_n(y') = z$. Then

$$g_n(y - y') = g_n(y) - g_n(y') = z - z = 0,$$

so, by exactness, there exists $u \in C'_n$ such that $y - y' = f_n(u)$. Suppose $x' \in C'_{n-1}$ is such that $f_{n-1}(x') = d_n(y')$. Then, choosing y', the definition of δ would lead to $\overline{x'}$. We have that

$$f_{n-1}(d'_n(u)) = d_n(f_n(u)) = d_n(y-y') = d_n(y)-d_n(y') = f_{n-1}(x)-f_{n-1}(x') = f_{n-1}(x-x').$$

Since f_{n-1} is an injection, this implies that $x - x' = d'_n(u)$. This means that in homology group $H_{n-1}(C')$ the classes $\overline{x}$ and $\overline{x'}$ are the same class. This concludes the proof of the fact that $\delta\colon Z_n(\overline{C}) \to \overline{x} \in H_{n-1}(C')$ is a well-defined mapping.

However, this is yet not the mapping we need. We have to show that δ "factors through" $B_n(\overline{C})$ i.e. that

$$\delta(z) = \delta(z')$$

if $\overline{z} = \overline{z'}$ in $H_n(\overline{C})$. This can be done in two ways. Classical way is to show that
(i) δ is a homomorphism of abelian groups,
(ii) $B_n(\overline{C}) \subset \operatorname{Ker}\delta$.
Then the standard application of Factorization Theorem of abelian groups (Proposition 7.8.) gives us what we need. Since in general theory development we need the fact that both δ and the induced mapping Δ are homomorphisms anyway, this is how this is done usually.
But since we are not asked to proof that Δ is a homomorphism, only that is well-defined, another, perhaps faster way is to show directly that

$$\delta(z) = \delta(z')$$

whenever $\overline{z} = \overline{z'}$ in $H_n(\overline{C})$. We'll show both ways.

Classical Way: We start by showing that δ is a homomorphism of abelian groups i.e. that

$$\delta(z + z') = \delta(z) + \delta(z')$$

for all $z, z' \in Z_n(\overline{C})$. We choose $y, y' \in C_n$ such that $g_n(y) = z$ and $g_n(y') = z'$. Then, by definition, $\delta(z) = \overline{x}$ and $\delta(z) = \overline{x'}$, where $x, x' \in Z_{n-1}(C')$ are chosen so that $f_{n-1}(x) = d_n(y)$ and $f_{n-1}(x') = d_n(y')$. Since f_{n-1}, d_n and g_n are homomorphisms we have that

$$g_n(y + y') = z + z',$$

$$f_{n-1}(x + x') = d_n(y + y').$$

Hence $x + x'$ qualifies as an element of $Z_{n-1}(C')$ such that

$$\delta(z + z') = \overline{x + x'} = \overline{x} + \overline{x'} = \delta(z) + \delta(z').$$

This proves that δ is a homomorphism of groups.
Next we show that $B_n(\overline{C}) \subset \operatorname{Ker}\delta$. Suppose $z = \overline{d}_{n+1}(w)$ for some $w \in \overline{C}_{n+1}$. Since g_{n+1} is a surjection there exists $v \in C_{n+1}$ such that $g_{n+1}(v) = w$. Then, since g is a chain mapping, we have that

$$g_n(d_{n+1}(v)) = \overline{d}_n(g(v)) = \overline{d}_n(w) = z.$$

Thus we can choose $y = d_{n+1}v$ to be the element of C_n with the property $g_n(y) = x$. Now $d_n(y) = d_n d_{n+1} v = 0$, so $\delta(x) = 0$, by constuction.

By Factorization theorem of abelian groups (Proposition 7.8.) there exists unique mapping $\Delta\colon H_n(\overline{C}) \to H_{n-1}(C')$ such that $\Delta(\overline{z}) = \delta(z)$ for all $z \in Z_n(\overline{C})$.

Direct way: We need to show that

$$\delta(z) = \delta(z')$$

whenever $\overline{z} = \overline{z'}$ in $H_n(\overline{C})$. If $\overline{z} = \overline{z'}$, then there exists $w \in \overline{C}_{n+1}$ such that $\overline{w} = z - z'$. Since g_{n+1} is a surjection there exists $v \in C_{n+1}$ such that $g_{n+1}(v) = w$. Then, since g is a chain mapping, we have that

$$g_n(d_{n+1}(v)) = \overline{d}_n(g(v)) = \overline{d}_n(w) = z - z'.$$

Choose $y \in C_n$ such that $g_n(y) = z$. Then for $y' = y + d_{n+1}(v)$ we have that $g_n(y') = z'$. Let $x \in C'_{n-1}$ be such that

$$f_{n-1}(x) = d_n(y).$$

Since $d_n(y') = d_n(y) + d_n d_{n+1}(v) = d_n(y)$, we also have that $f_{n-1}(x) = d_n(y')$. Hence

$$\delta(z) = \overline{x} = \delta(z')$$

and we are done. Indeed, it also seems like this proof is faster.

b) We need to show that $\operatorname{Im} f_* = \operatorname{Ker} g_*$. One direction is easy. Since the sequence (0.1) is exact, we have that $\operatorname{Im} f = \operatorname{Ker} g$, so in particular $\operatorname{Im} f \subset \operatorname{Ker} g$, which is the same thing as $g \circ f = 0$. Applying "star"-operator and its properties (Lemma 10.3.) (you can certainly assume those in exam situation) we get

$$g_* \circ f_* = (g \circ f)_* = 0_* = 0,$$

so $\operatorname{Im} f_* \subset \operatorname{Ker} g_*$. For another direction suppose $\overline{y} \in H_n(C)$ is such that $g_*(\overline{y}) = 0$. Here $y \in Z_n(C)$. Assumption means that $g_n(y) = \overline{d}_{n+1}(w)$ for some $w \in \overline{C}_{n+1}$. Let $v \in C_{n+1}$ be such that $g_{n+1}(v) = w$. Such an element exists, since g_{n+1} is a surjection. Since g is a chain mapping we have that

$$g_n(d_{n+1}v) = \overline{d}_{n+1}(g_{n+1}(v)) = \overline{d}_{n+1}(w) = g_n(y).$$

Hence $y - d_{n+1}v \in \operatorname{Ker} g = \operatorname{Im} f$, by exactness. Consequently, there exists $x \in C'_n$ such that

$$y - d_{n+1}v = f_n(x).$$

Since f_{n-1} is an injection, it follows easily that x is a cycle. Indeed

$$f_{n-1}(d'_n(x)) = d_n f_n(x) = d_n(y - d_{n+1}v) = 0,$$

so $d'_n(x) = 0$ by injectivity of f_{n-1}.

Thus there exists an equivalence class $\overline{x} \in H_n(C')$ and

$$f_*(\overline{x}) = \overline{f(x)} = \overline{y - d_{n+1}v} = \overline{y},$$

since the boundary element $d_{n+1}v$ becomes zero in homology.
We have shown that $\operatorname{Ker} g_* \subset \operatorname{Im} f_*$. This concludes the proof of the exactness at $H_n(C)$.

Grading:
a) $0,5$ points - for explaining the sheer algorithm that defines Δ, without proofs that is it is well defined,
$1,5$ points - for showing that δ is well-defined.
$1,5$ points - for showing that δ quotients through boundaries, hence produces well-defined Δ.
b) 1 point - for the proof of $\operatorname{Im} f_* \subset \operatorname{Ker} g_*$,
$1,5$-points - for the proof of $\operatorname{Ker} g_* \subset \operatorname{Im} f_*$.

2. a) Prove that $\widetilde{H}_m(S^n) \cong \widetilde{H}_{m-1}(S^{n-1})$ for all $n \geq 1$ and all $m \in \mathbb{Z}$ on the following conditions - you are allowed to use all properties of singular homology such as excision, homotopy axiom, long exact (reduced?) homology sequence of the pair, **except you are not** allowed to use Mayer-Vietoris sequence.
b) Now prove the claim of a) by using Mayer-Vietoris sequence (applying it in any situation we know it can be applied).

Solution: a) One way to do this is how we did it on pages 205-206 of the lecture material. First we take $A \subset U \subset S^n$ such that
(i) $\overline{A} \subset \operatorname{int} U$,
(ii) U is contractible,
(iii) The pair $(S^n \setminus A, U \setminus A)$ has the same homotopy type as the pair $(\overline{B}^n, \overline{B}^n \setminus \{0\})$.
In the lecture notes we choose

$$A = \{\mathbf{x} = (x_1, \ldots, x_{n+1}) \in S^n \mid x_{n+1} < 0\},$$

$$U = S^n \setminus \{e_{n+1}\}.$$

but, for example, any choice of the type

$$A = \{\mathbf{x} = (x_1, \ldots, x_{n+1}) \in S^n \mid x_{n+1} < t_1\},$$

$$U = \{\mathbf{x} = (x_1, \ldots, x_{n+1}) \in S^n \mid x_{n+1} < t_2\},$$

where $-1 < t_1 < t_2$ would do (and there also other possibilities). It is enough is that the properties (i)-(iii) above hold.

Suppose properties (i)-(iii) are true for A and U. From the long exact reduced homology sequence

$$\tilde{H}_m(U) = 0 \longrightarrow \tilde{H}_m(S^n) \xrightarrow{j_*} H_m(S^n, U) \longrightarrow \tilde{H}_{m-1}(U) = 0$$

we see that the homomorphism $j_*\colon \tilde{H}_m(S^n) \cong H_m(S^n, U)$ is an isomorphism, by exactness. Here $\tilde{H}_m(U) = 0$, because U is assumed contractible. This is a simple consequence (Corollary 13.17) of homotopy axiom (Proposition 13.1), which can be assumed known. Hence

$$\tilde{H}_m(S^n) \cong H_m(S^n, U),$$

for all $m \in \mathbb{Z}, n \geq 1$. Next we use excision (Theorem 14.1.) to excise the set A from the pair (S^n, U). Since the property (i) holds, by excision property we have that

$$H_m(S^n, U) \cong H_m(S^n \setminus A, U \setminus A).$$

By property (iii) and homotopy axiom (Proposition 13.1) we have that

$$H_m(S^n \setminus A, U \setminus A) \cong H_m(\overline{B}^n, S^{n-1}).$$

For example for the sets A and U chose above (and in lecture material) we have that the set $S^n \setminus A$ is a closed upper hemisphere

$$\{\mathbf{x} \in S^n | x_{n+1} \geq 0\}$$

of the sphere S^n, which is homeomorphic to the closed ball $\overline{B}^n$, this fact was proved in the course and used a lot, so one can assume it known. Moreover, under this homeomorphism the subset $U \setminus A$ corresponds to the punctured ball $\overline{B}^n \setminus \{0\}$. Hence

$$H_m(S^n \setminus A, U \setminus A) \cong H_m(\overline{B}^n, \overline{B}^n \setminus \{0\}).$$

For this choice the pairs $(S^n \setminus A, U \setminus A)$ and $(\overline{B}^n, \overline{B}^n \setminus \{0\})$ are even homeomorphic, but it would be enough that they have the same homotopy type, by homotopy axiom.

Now, the space $\overline{B}^n$ is contractible, so its reduced homology groups are trivial. The part of the long exact reduced homology sequence of the pair $(\overline{B}^n, \overline{B}^n \setminus \{0\})$ is the sequence

$$\tilde{H}_m(\overline{B}^n) = 0 \longrightarrow H_m(\overline{B}^n, \overline{B}^n \setminus \{0\}) \xrightarrow{\Delta} \tilde{H}_{m-1}(\overline{B}^n \setminus \{0\}) \longrightarrow \tilde{H}_{m-1}(\overline{B}^n) = 0.$$

By exactness Δ is an isomorphism, hence

$$H_m(\overline{B}^n, \overline{B}^n \setminus \{0\}) \cong \widetilde{H}_{m-1}(\overline{B}^n \setminus \{0\})$$

for all $m \in \mathbb{Z}$.

Finally we use homotopy axiom. It is well-known (i.e. can be assumed known) that $\overline{B}^n \setminus \{0\}$ has the same homotopy type as the sphere S^{n-1}. Hence

$$\widetilde{H}_{m-1}(\overline{B}^n \setminus \{0\}) \cong \widetilde{H}_{m-1}(S^{n-1}).$$

Putting all the isomorphisms together we have proved that for all $m \in \mathbb{Z}$ and all $n \geq 1$ the claim

$$\widetilde{H}_m(S^n) \cong \widetilde{H}_{m-1}(S^{n-1})$$

holds.

Remarks: 1) Of course this is not the only way to do the claim right. For example in Exercise 11.4. we have proved the following claim using excision and homotopy axiom, but not Mayer-Vietoris:
The inclusions of pairs $(B_+, S^{n-1}) \to (S^n, B_-)$ and $(B_-, S^{n-1}) \to (S^n, B_+)$ induce isomorphisms in homology for all dimensions. Here

$$B_+ = \{x \in S^n \mid x_{n+1} \geq 0\}$$

and

$$B_- = \{x \in S^n \mid x_{n+1} \leq 0\}.$$

Since both B_- and B_+ are contractible (being homeomorphic to $\overline{B}^n$), using long exact reduced homology sequences we obtain

$$\widetilde{H}_m(S^n) \cong H_m(S^n, B_-) \cong H_m((B_+, S^{n-1}) \cong \widetilde{H}_{m-1}(S^{n-1}).$$

For the details see the official Solutions of Exercise 11.4.

2) Since the claim is for reduced groups, it is wise to use reduced long exact sequence, not ordinary one. In fact, it is almost always wise to use reduced long exact sequence. With ordinary long exact sequence we have to deal with exceptional cases separately and the calculations become long, difficult and tedious.

b) Using Mayer-Vietoris it is enough to come up with the proper triad $(S^n; U, V)$ such that
(i) U and V are both contractible,
(ii) $U \cap V$ has the homotopy type of S^{n-1}.

Then we have **reduced** Mayer-Vietoris sequence

$$\ldots \longrightarrow \widetilde{H}_m(U) \oplus \widetilde{H}_m(V) \longrightarrow \widetilde{H}_m(S^n) \xrightarrow{\Delta} \widetilde{H}_{m-1}(U \cap V) \longrightarrow \widetilde{H}_{m-1}(U) \oplus \widetilde{H}_{m-1}(V) \longrightarrow \ldots,$$

in which $\widetilde{H}_m(U) \oplus \widetilde{H}_m(V) = 0$ for all $m \in \mathbb{Z}$ and $\widetilde{H}_{m-1}(U \cap V) = \widetilde{H}_{m-1}(S^{n-1})$. The claim then follows by exactness. The only thing that is missing from this extremely simple solution is an example of the proper triad $(S^n; U, V)$ that satisfies (i) and (ii). One possibility is to take

$$U = S^n \setminus \{\mathbf{e}_{n+1}\},$$

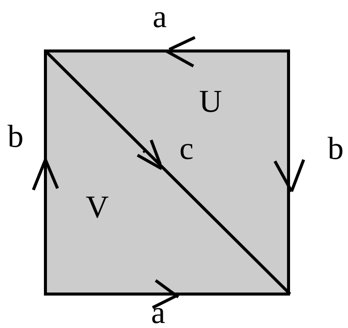
$$V = S^n \setminus \{-\mathbf{e}_{n+1}\}$$

Then $\operatorname{int} U \cup \operatorname{int} V = S^n$, so the triad is proper by Lemma 16.6.
Even simpler alternative is to use Δ-complex-approach. Indeed by Proposition 16.12 if K is a Δ-complex and L_1 and L_2 are subcomplexes of K such that $K = L_1 \cup L_2$, then $(|K|; |L_1|, |L_2|)$ is a proper triad. There exist triangulation K of S^n such that $(|K|; |L_1|, |L_2|)$, where L_1 and L_2 are subcomplexes of K such that $K = L_1 \cup L_2$, is homeomorphic to $(S^n; U, V)$. For instance it can be achieved simply by taking two ordered n-simplices σ_1 and σ_2 and identify all their corresponding faces. Then $|L|_1 \cap |L_2|$ is even homeomorphic to S^{n-1}. This is the way it is done in the Example 16.13.

It is also possible to use ordinary Mayer-Vietoris instead of reduced, but, as usual, it leads to much more complicated calculations. This way is illustrated in Example 16.7.

Grading:
a) 3 points, b) 3 points.

3. Suppose K is a 2-dimensional Δ-complex represented by the schematic picture below and let L be the subcomplex consisting of the 1-simplices a and b and their vertices.

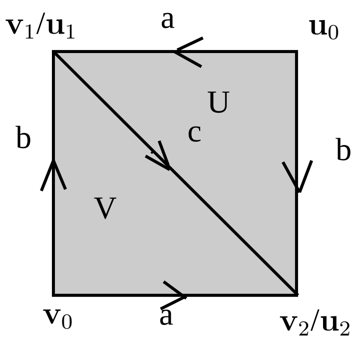


a) Calculate simplicial homology groups $H_n(K)$, $H_n(L)$ and $H_n(K, L)$ for all $n \in \mathbb{Z}$, by definition.
b) Investigate how the mappings Δ_2, i_* and j_* from the following portion of the long exact homology sequence

$$\ldots \longrightarrow H_2(K,L) \xrightarrow{\Delta_2} H_1(L) \xrightarrow{i_*} H_1(K) \xrightarrow{j_*} H_1(K,L) \longrightarrow \ldots$$

of the pair (K, L) are defined (in terms of elements of groups).

Solution: Notice that the polyhedron of K is the projective plane $\mathbb{R}P^2$. We start by ordering the complex, i.e. putting the order on the vertices of both triangles. Here is one way:



The complex K has two geometrical 2-simplices, $U = [\mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2]$ and $V = [\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2]$, three 1-dimensional simplices a, b, c. This forces the following identifications on vertices:

$$\mathbf{v}_0 = \mathbf{u}_0 = x, \mathbf{v}_1 = \mathbf{v}_2 = \mathbf{u}_1 = \mathbf{u}_2 = y.$$

Hence there are two 0-simplices.

The complex L, thus, has two 1-simplices a and b and two 0-simplices x, y.

a) **Homology of** K.

Since K has no simplices in dimensions other than $0, 1, 2$, we automatically have that $H_n(K) = 0$ for $k \neq 0, 1, 2$. Now $C_2(K) = \mathbb{Z}[U] \oplus \mathbb{Z}[V]$ and

$$d_2(nU + mV) = n(c - b + a) + m(c - a + b) = (n - m)a + (m - n)b + (n + m)c = 0$$

if and only if $n + m = n - m = 0$ i.e. if and only if $n = m = 0$. Hence d_2 is injective, so its kernel is trivial, and consequently the group $H_2(K)$ is trivial.
The very same calculation shows that $B_1(K) = \operatorname{Im} d_2$ is a group generated by the elements $c - b + a$ and $c - a + b$, which form the **free** basis of $B_1(K)$.
Next we calculate d_1. Since $C_1(K) = \mathbb{Z}[a] \oplus \mathbb{Z}[b] \oplus \mathbb{Z}[c]$, we have that

$$d_1(na + mb + lc) = n(y - x) + m(y - x) = (n + m)(y - x) = 0$$

if and only if $n = -m$. Thus

$$Z_1(K) = \operatorname{Ker} d_1 = \{n(a - b) + lc | n, l \in \mathbb{Z}\},$$

so $Z_1(K)$ is a free group generated by the elements $a - b$ and c. Obviously $\{a - b, c\}$ is a free set, hence is a basis of $Z_1(K)$.

In what follows we need the standard "switching of basis "-trick, which is bases on the following fact (proved in Exercise 7.2.). Suppose $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is a basis of a free abelian group G. Then also $\{\alpha_1 \pm \alpha_2, \alpha_2, \ldots, \alpha_n\}$ is a basis of the group G. This fact can certainly be assumed known.

We apply this fact first to the basis $\{c - b + a, c - a + b\}$ of $B_1(K)$ to obtain the new basis

$$\{2c = (c - b + a) + (c - a + b), c - (a - b)\}$$

for $B_1(K)$. On the other hand, the same fact applied to the basis $\{c, a-b\}$ of $Z_1(K)$ gives the basis $\{c, c-(a-b)\}$ for $Z_1(K)$. Hence

$$H_1(K) = (\mathbb{Z}[c] \oplus \mathbb{Z}[c-(a-b)])/(\mathbb{Z}[2c] \oplus \mathbb{Z}[c-(a-b)] \cong \mathbb{Z}[c]/\mathbb{Z}[2c] \cong \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}_2.$$

Precisely put $H_1(K) = \{0, [c]\}$.

For 0-homology we have that $H_0(K) = (\mathbb{Z}[x] \oplus \mathbb{Z}[y])/B_0(K)$. Since

$$d_1(na + mb + lc) = (n+m)(y-x),$$

we see that $B_0(K) = \mathbb{Z}[y-x]$. Since $\{y-x, y\}$ is a basis for $C_0(K)$ (by change of basis again), we see that

$$H_0(K)(\mathbb{Z}[y-x] \oplus \mathbb{Z}[y])/\mathbb{Z}[y-x] \cong \mathbb{Z}[y] \cong \mathbb{Z}.$$

Homology of L.

Since K has no simplices in dimensions other than $0, 1$, we automatically have that $H_n(K) = 0$ for $k \neq 0, 1$. We have that

$$d_1(na + mb) = (n+m)(y-x),$$

so $H_1(L) \cong Z_1(L) = \mathbb{Z}[a-b] \cong \mathbb{Z}$. Also this shows that $B_0(L) = \mathbb{Z}[y-x]$, while

$$Z_0(L) = C_0(L) = \mathbb{Z}[x] \oplus \mathbb{Z}[y] = \mathbb{Z}[x-y] \oplus \mathbb{Z}[y]$$

(switching of basis!). Hence

$$H_0(L) = (\mathbb{Z}[x-y] \oplus \mathbb{Z}[y])/\mathbb{Z}[x-y] \cong \mathbb{Z}[y] \cong \mathbb{Z}.$$

Homology of (K, L).

We have that

$$C_2(K,L) = \mathbb{Z}[U] \oplus \mathbb{Z}[V],$$
$$C_1(K,L) = (\mathbb{Z}[a] \oplus \mathbb{Z}[b] \oplus \mathbb{Z}[c])/(\mathbb{Z}[a] \oplus \mathbb{Z}[b] = \mathbb{Z}[c],$$
$$C_0(K,L) = (\mathbb{Z}[x] \oplus \mathbb{Z}[y])/(\mathbb{Z}[x] \oplus \mathbb{Z}[y]) = 0,$$

and all other groups $C_n(K)$ are trivially zero. Hence only $H_2(K,L)$ and $H_1(K,L)$ require actual calculation.
Notice that for convenience we notate the equivalence classes as the elements, for example the class of c is denoted simply by c. In the complex $C(K,L)$ we have that $d_2(U) = c = d_1(V)$, so

$$d_2(nU + mV) = (n+m)c.$$

This shows that $H_2(K,L) \cong Z_2(K,L) = \mathbb{Z}[U-V] \cong \mathbb{Z}$ and $B_1(K,L) = \mathbb{Z}[c] = C_1(K,L)$. This can only mean that $Z_1(K,L) = B_1(K,L) = C_1(K,L)$, so $H_1(K,L) =$

0. In the end, the only non-trivial homology group of (K, L) is $H_2(K, L) \cong \mathbb{Z}$.

b) By the result of a) the sequence

$$\dots \longrightarrow H_2(K, L) \xrightarrow{\Delta_2} H_1(L) \xrightarrow{i_*} H_1(K) \xrightarrow{j_*} H_1(K, L) \longrightarrow \dots$$

up to isomorpisms looks like the sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \xrightarrow{\beta} \mathbb{Z}_2 \longrightarrow 0 \longrightarrow \dots$$

In particular $j_* = 0$, since $H_1(K, L) = 0$.
By exactness $\beta\colon \mathbb{Z} \to \mathbb{Z}_2$ is surjective. Since $\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$, this means that $\beta(1) = \bar{1}$, since otherwise $\beta(1) = 0$, which implies that $\beta(n) = n\beta(1) = 0$ for all $n \in \mathbb{Z}$ and β cannot be surjective. Hence $\beta = \mathrm{pr}$ is nothing but a projection to the quotient group. For the groups of original sequence we had $H_1(K) = \mathbb{Z}[a - b]$ and $H_1(K, L) = \{0, [c]\}$, so $i_*([a - b]) = c$, more generally $i_*(n[a - b]) = n[c]$. This can also be proved directly by calculating. All we need to show is that in $H_1(K)$ we have

$$[a - b] = [c].$$

This follows from the fact in $C(K)$ we have

$$d_1(V) = c - a + b = c - (a - b),$$

so $c - (a - b)$ is a boundary element.

It remains to calculate Δ_2. Again, its isomorphic version is the mapping $\alpha\colon \mathbb{Z} \to \mathbb{Z}$ which, by exactness, is injective and $\operatorname{Im}\alpha = \operatorname{Ker}\beta = 2\mathbb{Z}$. There are exactly two homomorphisms that satisfy this condition - the mapping given by $\alpha(n) = 2n$ or by $\alpha(n) = 2n$. In terms of original groups we have that $H_2(K, L) = \mathbb{Z}[U - V]$ and $H_1(L) = \mathbb{Z}[a - b]$, so this means that either $\Delta([U - V]) = 2[a - b]$ or $\Delta([U - V]) = -2[a - b] = 2[b - a]$. In order to decide which one it is, this time we actually need to calculate it directly. We take a cycle $U - V \in C_2(K, L)$. Then obviously $U - V \in C_2(K)$ is an element with $j_2(U - V) = U - V$. We have that

$$d_2(U - V) = (c - b + a) - (c - a + b) = 2a - 2b = 2(a - b).$$

Hence, by definition of Δ, $\Delta([U - V]) = 2[a - b]$. This concludes the proof.

Grading:
1 point for groups $H_n(K)$,
1 point for groups $H_n(L)$,
1 point for groups $H_n(K, L)$,
$0,5$ point for j_*,
1 point for i_*,
$1,5$ points for Δ.

4. Suppose $C \subset S^n$ is homeomorphic to S^k, for $0 \leq k \leq n-1$. Prove that

$$\widetilde{H}_m(S^n \setminus C) = \begin{cases} 0, & \text{for } m \neq n-k-1, \\ \mathbb{Z}, & \text{for } m = n-k-1. \end{cases}$$

You can assume and freely use the following fact: Suppose $B \subset S^n$ is homeomorphic to $\overline{B}^k$, for $0 \leq k \leq n$. Then $\widetilde{H}_m(S^n \setminus B) = 0$ for all $m \in \mathbb{Z}$.

Where in your proof you use the assumption $k \leq n-1$? Do we need that restriction, for instance would the claim still be true for $k = n$? Explain.

Solution: Except for the extra question in the end this is just the proof of Lemma 17.4. Here is straightforward copypasted proof of this Lemma from the lecture material:

Proof by induction on k.

If $k = 0$, then $C = \{a, b\}$ is space consisting of two isolated points. Since S^n minus a point is homeomorphic to $\mathbb{R}^n$, S^n minus two points is homeomorphic to $\mathbb{R}^n$ minus a point, i.e. essentially homeomorphic to the space $\mathbb{R}^n \setminus \{\mathbf{0}\}$. This space, on the other hand, has a homotopy type of the sphere S^{n-1}. Hence in this case

$$\widetilde{H}_m(S^n \setminus C) = \begin{cases} 0, & \text{for } m \neq n-1, \\ \mathbb{Z}, & \text{for } m = n-1, \end{cases}$$

which is exactly the claim for $k = 0$.

Suppose claim is true for $k - 1 \geq 0$ and let us show it for k. Let $f\colon S^k \to C$ be a homeomorphism. Denote $C_1 = f(S_+)$ and $C_2 = f(S_-)$, where S_+, S_- are upper and lower hemisphere of S^k, as usual. Since hemispheres are compact, the sets C_1, C_2 are compact, hence closed in Hausdorff space S^n. It follows that their complements are open. Hence $\{S^n \setminus C_1, S^n \setminus C_2\}$ is an open covering of the space $S^n \setminus (C_1 \cap C_2) = S^n \setminus D$, where $D = C_1 \cap C_2$ is homeomorphic to the sphere S^{k-1}. Also

$$(S^n \setminus C_1) \cap (S^n \setminus C_2) = S^n \setminus C.$$

Since S_+ and S_- are both homeomorphic to the closed ball $\overline{B}^k$, the same is true for subsets C_1 and C_2. Hence, by the result we are allowed to use, both spaces $S^n \setminus C_1$ and $S^n \setminus C_2$ have trivial reduced groups in all dimensions. The reduced Mayer-Vietoris sequence of the proper triad $(S^n \setminus C; S^n \setminus C_1, S^n \setminus C_2)$, implies, in the usual way, that $\Delta\colon \widetilde{H}_{m+1}(S^n \setminus D) \to \widetilde{H}_m(S^n \setminus C)$ is an isomorphism for all $m \in \mathbb{N}$. Since, by inductive assumption,

$$\widetilde{H}_m(S^n \setminus C) = \begin{cases} 0, & \text{for } m \neq n-k, \\ \mathbb{Z}, & \text{for } m = n-k, \end{cases}$$

it follows that

$$\widetilde{H}_m(S^n \setminus B) = \begin{cases} 0, & \text{for } m \neq n-k-1, \\ \mathbb{Z}, & \text{for } m = n-k-1. \end{cases}$$

Now, that us also comment on the extra question. Where exactly the assumption $k \leq n-1$ is used above? First of all, the proof claims to be **the proof by induction** on k. Now, isn't induction usually something that should work for all natural numbers? If the initial step and inductive steps are universally true, the claim also should be true for all $n \in \mathbb{N}$.
But let us take a look at the claim for $k = n$. Then C is a subset of S^n homeomorphic to S^n. But this is possible only when $C = S^n$. We actually know it from the theory of manifolds - any embedding $f\colon C \to N$ between compact n-manifold C and connected n-manifold N must be bijective (Corollary 17.10). Hence when $k = n$ we have that $C = S^n$ and hence $S^n \setminus C$ is an empty space. For an empty space we did not define reduced groups, so the claim do not even make sense for case $k = n$.
Notice that in order to prove Corollary 17.10 we need invariance of domain which we proved using the claim of Lemma 17.4, so we cannot use it before it is proved!

The observation about reduced groups not being defined for empty space is actually the key to the extra question. We used reduced groups throughout the proof, but when one uses reduced groups one must be careful - they are not defined for empty space.
What about induction? Well, induction is often used to proof claims about all natural numbers, but that is not the only application. By induction one can also prove claims concerning finite ordered sets - for example interval $[0, N]$ for some natural N. Then it is enough to prove that the claim is true for 0 (initial step) and to prove that if $k < N$ and the claim is true for k, then it is also true for $k+1$. This is exactly what we did above - for $N = n - 1$.

Remark: It is actually possible to extend the notion of reduced groups to emptyset. Then $\widetilde{H}_{-1}(\emptyset) = \mathbb{Z}$, so the claim of this exercise becomes formally true for the case $k = n$ as well!

Grading:
$1,5$ points for initial step
$3,5$ points for inductive step
1 point for some satisfactory answer to the extra question.